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On the Primitive Groups of Transformations in Space of Four Dimensions.

By James M. Page.

In a series of papers published at various intervals since 1873 Sophus Lie has developed a new mathematical theory, which he calls the "Theory of Groups of Transformations." His researches are nearly related to several other branches of mathematics, especially to the "Theory of Substitutions" of Galois, to the Modern Geometry and Modern Algebra, and to the Theory of Differential Equations. Lie has very materially modified this last-named theory by basing it upon his Theory of Groups, and by showing that this is the natural and correct starting point for a Theory of Differential Equations.

But, in order to derive practical advantage from the Theory of Groups for the Theory of Differential Equations and its kindred branches of mathematics, it is necessary to know some, and if possible, *all* groups of transformations. Lie has already given methods for finding all of certain classes of groups in any number of variables, although to carry out the necessary calculations in some cases may be practically impossible. However, he has actually calculated all groups in one, two, and three variables. It will be the object of this paper to find all of a certain class of groups—the so-called *primitive* ones—in *four* variables.

As Lie's theory, since he has not yet published a connected work on the subject, is not generally known, we shall collect the principal definitions and theorems of the same needed in this paper, in an

Introduction.

A. We shall always operate with "infinitesimal point transformations," inasmuch as they are most convenient for our calculations, and Lie has shown how to find the groups of finite transformations by very simple means, when we know the groups of infinitesimal transformations.

Lie defines an infinitesimal point transformation in n variables by the equations

$$x_i' = x_i + \xi_i(x_1 \ldots . x_n)\,\delta t, \qquad i = 1 \ldots . n, \qquad (\alpha)$$

where δt is an infinitesimal quantity, and where ξ_i are "analytical functions" of their arguments, in Weierstrass' sense of the word.

For the infinitesimal transformation (α), Lie has introduced the symbol

$$X(f) \equiv \sum_1^n {}^i\, \xi_i(x_1 \ldots . x_n)\,\frac{\partial f}{\partial x_i},$$

or, when we write as is customary,

$$p_i \equiv \frac{\partial f}{\partial x_i},$$

the symbol is

$$X(f) \equiv \sum_1^n {}^i\, \xi_i(x_1 \ldots . x_n)\,p_i.$$

From this symbol we can see at a glance the form of (α); for if f is any function of $x_1 \ldots . x_n$, we have

$$f(x_1' \ldots . x_n') = f(x_1 \ldots . x_n) + X(f)\,\delta t,$$

from which follows

$$x_i' = x_i + \xi_i(x_1 \ldots . x_n)\,\delta t.$$

It is easy to see that the symbol $X(f)$, or as we shall for convenience simply write it, Xf, is independent of the choice of variables.*

If

$$X_1 f,\ X_2 f \ldots . X_r f$$

are r infinitesimal transformations, they are said to be *independent*, when it is impossible to choose r such constants c_k as make the expression

$$c_1 X_1 f + c_2 X_2 f + \ldots . + c_r X_r f$$

vanish identically.

When we perform the operation

$$X_i(X_k(f)) - X_k(X_i(f)) \equiv \sum_1^n {}^j\, \{X_i(\xi_{jk}) - X_k(\xi_{ji})\}\,\frac{\partial f}{\partial x_j},$$

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*If the infinitesimal transformation Xf is repeated an infinite number of times, we get ∞^1 finite transformations

$$x_i' = x_i + t . X(x_i) + \frac{t^2}{1.2} . XX(x_i) + \ldots . ,$$

where t is an arbitrary parameter.

we say that we *combine* the two infinitesimal transformations $X_i f$ and $X_k f$. For this operation we shall write Jacobi's simple symbol

$$(X_i X_k).$$

Now Lie has given the fundamental

Definition: "If r independent infinitesimal transformations $X_1 f \dots\dots X_r f$ satisfy in pairs all relations of the form

$$(X_i X_\kappa) \equiv \sum_{1}^{r}{}^{s}\, c_{iks} X_s(f), \qquad i,\, k = 1 \dots\dots r,$$

where the c_{iks} are constants, then form the infinitesimal transformations

$$\sum_{1}^{r}{}^{k}\, c_k X_k f,$$

where the c_k are constants, a group with r members."

The system of constants c_{iks} is called the *composition* (Zusammensetzung) of our group.

Any three transformations $X_i f$, $X_j f$, $X_k f$ must satisfy the celebrated identity of Jacobi,

$$(X_i(X_j X_k)) + (X_j(X_k X_i)) + (X_k(X_j X_i)) \equiv 0,$$

and this gives relations which the c_{iks} must satisfy, if they form the composition of a group, viz:

$$\sum_{1}^{r}{}^{s}\, (c_{ijs} . c_{sk\nu} + c_{jks} . c_{is\nu} + c_{kis} . c_{sj\nu}) = 0,$$

$$(i,\, k,\, j,\, \nu = 1 \dots\dots r).$$

Lie has shown, conversely, that if we have a system of c_{iks} which satisfy the equations last written, there are always groups with r members which possess this composition.

If all $$c_{iks} = 0,$$

we say that our above transformations are *in pairs commutable* (paarweise vertauschbar), or simply *commutable.* We may take any r independent infinitesimal transformations of our above group, say $X_1 f \dots\dots X_r f$, as representatives of that group, since all of its infinitesimal transformations are included in the general infinitesimal transformation of the group

$$\sum_{1}^{r}{}^{k}\, c'_k X_k f, \quad (c'_k \text{ const.}).$$

and when we know $X_1 f \dots X_r f$, we may at once write all other transformations of the group.

For a group of transformations of r members we shall sometimes use the symbol G_r.

If among all the infinitesimal transformations

$$\sum_1^r {}^k \, c_k X_k f$$

of our G_r, there are some, say

$$X_1 f \dots X_{r-\rho} f,$$

which form themselves a group, we call this group a *subgroup* (Untergruppe) of the original G_r. If the G_r contains a subgroup

$$Y_k f \equiv d_{k1} X_1 f + \dots + d_{kr} X_r f, \qquad k = 1 \dots r - \rho,$$

where relations hold of form

$$(Y_k X_i) \equiv \sum_1^{r-\rho} {}^s \, d_{iks} Y_s f, \qquad i, k = 1 \dots r,$$

then is the subgroup $Y_1 f \dots Y_{r-\rho} f$ said to be *invariant* in the G_r $X_1 f \dots X_r f$; and the $Y_k f$ are said to form an *invariant subgroup.*

The finding of all subgroups of a given G_r involves only the performance of algebraic operations.

Two groups are said to be *similar* (ähnlich) when by a proper choice of independent variables the one group can be transformed to the other; that is, if we have the two G_r,

$$X_1 f \dots X_r f \text{ and } X_1' f \dots X_r' f,$$

they are *similar*, when it is possible to introduce into $X_k f$ such new independent variables that relations hold of the forms

$$X_k f = \sum_1^r {}^i \, c_{ki} X_i' f, \qquad k = 1 \dots r,$$

where c_{ki} are constants.

We shall consider all groups *known*, which are similar to a given group.

B. For a space of n dimensions we shall use the customary symbol R_n; and for a manifoldness of k dimensions, M_k.

In all operations with infinitesimal transformations we shall be satisfied with a *first approximation* to accuracy; that is, we shall drop out of consideration infinitesimals of the second and higher orders.

Now a function
$$\phi\,(x_1 \ldots . x_n)$$
is transformed by the infinitesimal transformation Xf, in the same variables, to
$$\phi + X(\phi)\,\delta t.$$
We say that the function ϕ *admits of* (gestattet) the infinitesimal transformation Xf, or is *invariant under* Xf, when
$$X(\phi) \equiv 0,$$
i. e. when ϕ is a solution of the linear partial differential equation of the first order,
$$Xf = 0.$$
If in the ordinary XY-plane, a function $\phi(xy)$ admits of each infinitesimal transformation of a G_r, it is evident that each of the ∞^1 curves
$$\phi\,(xy) = \text{const.}$$
is unchanged by the transformations of the group; that is, each curve of the family is *invariant* under the group. Thus each ordinary point of the plane must be simply moved up on one of the curves by the transformations of the group. In this case the G_r is said to be *intransitive.* If, on the contrary, a G_r in xy can transform every ordinary point of the plane to every other ordinary point of the plane, the G_r is said to be *transitive.*

Similarly in R_n, if a function $\phi\,(x_1 \ldots . x_n)$ admits of every infinitesimal transformation of a group, it is evident that each of the ∞^1 M_{n-1},
$$\phi\,(x_1 \ldots . x_n) = \text{const.}$$
is unchanged by the transformations of the G_r, is *invariant* under the G_r. Here every point of general position in R_n is moved up on one of these M_{n-1} by the transformations of our G_r. In this case our group in n variables is said to be *intransitive.*

If, on the contrary, every ordinary point in R_n can be carried by the transformations of our G_r to every other such point in R_n, we call the group *transitive.* Analytically expressed, the group
$$X_1 f \ldots . X_r f$$
is *transitive*, when the equations
$$X_1 f = 0 \ldots . X_r f = 0$$
have no common solution.

But each member of the family of curves

$$\phi(xy) = \text{const.}$$

in the plane, or, as we may write it,

$$\Psi(\phi) = \text{const.},$$

where Ψ is an arbitrary function of ϕ, need not be invariant for the whole family to be invariant. The family as a whole is still invariant when its various members are swapped around among each other by the transformations of our G_r. Now we know

$$\phi(xy) = \text{const.}$$

goes by means of the infinitesimal transformation Xf to

$$\phi + X(\phi)\,\delta t = \text{const.}$$

Thus if

$$X(\phi) \equiv \Psi(\phi)$$

the family of curves is as a whole evidently invariant.

This family of curves is defined by a linear partial differential equation IO,

$$Af \equiv X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0,$$

and when the family is invariant under Xf, the differential equation is also said to be invariant under that transformation.

The analytical criterium that

$$Af = 0$$

should be invariant under Xf is

$$(X, A) \equiv \rho(xy).Af,$$

where ρ is some function of x and y.

Now a group

$$X_1f \ldots . X_rf$$

in the plane is said to be *imprimitive* when it leaves a family of ∞^1 curves,

$$\phi(xy) = c$$

invariant, *i. e.* when

$$X_i(\phi) = \Psi_i(\phi), \qquad i = 1 \ldots . r.$$

Otherwise is the group said to be *primitive.*

Similarly, in R_n the $\infty^{n-q}M_q$,

$$\phi_1(x_1 \ldots . x_n) = c_1, \ldots . \phi_{n-q}(x_1 \ldots . x_n) = c_{n-q}$$

are said to be invariant as a whole under the transformation Xf, in the same variables, when

$$X(\phi_i) = \Omega_i(\phi_1 \ldots . \phi_{n-q}),$$

where the Ω are some functions of $\phi_1 \ldots . \phi_{n-q}$. These M_q are represented in R_n by a system of q linear partial differential equations *IO*,

$$Y_1f = 0, \ldots . Y_qf = 0,$$

where

$$Y_kf \equiv \sum_1^n {}^i \chi_{ki}(x_1 \ldots . x_n) \frac{\partial f}{\partial x_i};$$

and for these Y_kf must hold the relations

$$(Y_iY_j) = \sum_1^q {}^s \psi_{ijs}(x_1 \ldots . x_n) Y_sf.$$

Such a system is called a "complete (vollständig)* system with q members."

It may be shown that a complete system with q members in n variables always has $(n - q)$ independent solutions.

The analytical criterium that this system should be invariant under the transformation Xf in the same variables is

$$(X, Y_i) = \sum_1^q {}^s \rho_{is}(x_1 \ldots . x_n) Y_sf,$$

where the ρ are some functions of $x_1 \ldots . x_n$.

Now we say that a group $X_1f \ldots . X_rf$ in n variables is *imprimitive*, when it leaves a family of $\infty^{n-q}M_q$ (which fill R_n exactly once),

$$\phi_1(x_1 \ldots . x_n) = c_1 \ldots . \phi_{n-q}(x_1 \ldots . x_n) = c_{n-q}$$

invariant, *i. e.* when

$$X_i(\phi_k) = \Omega_{ki}(\phi_1 \ldots . \phi_{n-q}), \quad \begin{cases} k = 1 \ldots . n - q \\ i = 1 \ldots . r \end{cases},$$

where the Ω are some functions of $\phi_1 \ldots . \phi_{n-q}$. Otherwise is the group said to be *primitive.*

If we hold any ordinary point in the plane, the directions through it are transformed among each other by the transformations of any group. If now the group leaves a family of ∞^1 curves,

$$\phi(xy) = c$$

invariant, one such curve goes through each ordinary point in the plane; and when we hold such a point, the curve, and its tangential direction through that point, must remain invariant with the point. Lie has shown, *vice versa*, that a group in the plane is *imprimitive*, when with each ordinary point which we hold,

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*The *Theory of the Complete System* was developed by Jacobi and Clebsch.

an invariant direction through the point is connected—a necessary and sufficient criterium.

Thus, a group in the plane is *primitive* when with each ordinary point which we hold, no invariant direction is connected.

Similarly in R_n, if we have a group which leaves a family of $\infty^{n-q}M_q$ invariant, through each ordinary point goes a pencil of ∞^{q-1} directions, which are transformed among each other by the transformations of the group, when we hold the point.

Lie has shown that if with each ordinary point in R_n such a pencil of ∞^{q-1} directions is invariantly connected, a system of q linear partial differential equations IO is invariant. If this system is *complete*, then it is evident that the group in question is imprimitive; if the system is not complete, the group *may* be primitive.

Applying the above to $n = 4$, we see that in R_4 there are *three* cases: A single direction may be invariant with each ordinary point; or a pencil of ∞^2 directions, and no single direction, may be invariant; or a pencil of ∞^1 directions—and no single direction, and no pencil of ∞^1 directions—may be invariant. In the first case the groups are evidently imprimitive; at any rate, it is easy to see that they must be so. Lie has shown that they are also in the second case imprimitive. In the third case, however, the groups *may* be primitive.

C. From now on we shall confine ourselves to four variables.

In the infinitesimal transformation

$$Xf \equiv \sum_{1}^{4} {}^{i}\, \xi_i (x_1 \ldots . x_4)\, p_i$$

the ξ_i are analytical functions. Thus, if x_i^0 is an ordinary point in R_4, the ξ_i may be expanded in its neighborhood in powers of $x_i - x_i^0$. Let us choose the origin as an ordinary point, and then we may expand the ξ_i in its neighborhood in powers of x_i. Thus we may write for the neighborhood of the origin

$$Xf \equiv \sum_{1}^{4} {}^{i}\, \alpha_i p_i + \sum_{ck}^{1\ldots 4} \beta_{ik} x_i p_k + \sum_{ikj}^{1\ldots 4} \gamma_{ikj} x_i x_k p_j + \ldots .$$

Here α, β .... are constants, some of which may be zero. If the constant coefficients of all terms up to the $(k-1)^{\text{st}}$ vanish, that is, if the transformation Xf begin with terms of the k^{th} degree, we say *Xf is of the k^{th} order.* We call the terms of the k^{th} degree of a transformation of the k^{th} order the *initial terms* of the transformation.

It is easy to see that a transformation of the i^{th} order combined with a transformation of the k^{th} order gives a transformation of at least the $(i+k-1)$ order.

We may remark that all transformations of the forms $\sum_{1}^{4} {}_i\, a_i p_i$ are *mere translations*, and $\sum_{ik}^{1\dots 4} b_{ik} x_i p_k$ *linear homogeneous transformations.*

Our object is to find all groups in R_4 which transform the directions through an ordinary point, which we hold, so that no direction and no plane pencil of ∞^2 directions is invariant. Having chosen the origin as an ordinary point, let us hold it, and see how the directions through it are transformed by a transformation Xf. If we expand Xf in the neighborhood of the origin, we see that the most general transformation which leaves this point invariant has the form

$$\overline{X}f \equiv \sum_{ik}^{1\dots 4} \beta_{ik} x_i p_k + \sum_{ikj}^{1\dots 4} \gamma_{\,kj} x_i x_k p_j + \dots.$$

But $\overline{X}f$ does transform the points, very near the origin, with the coordinates dx_i. Any such point dx_i^0, together with the origin, determines a direction through the origin; and these directions are transformed just as the points very near the origin dx_i are. But we see, when we drop infinitesimals of the second and higher orders, that $\overline{X}f$ transforms the points dx_i, that is, the directions through the origin, just as

$$Yf \equiv \sum_{ik}^{1\dots 4} \beta_{ik} x_i p_k$$

does; that is, the directions through the origin are transformed by a linear homogeneous transformation. If now we have a G_r,

$$X_1 f \dots . X_r f,$$

Lie has proved that all transformations of the G_r which leave a point invariant, form a subgroup of the original G_r. We have seen that all primitive groups are also transitive. Thus in the G_r which we seek there must always be four infinitesimal transformations of the zero order:

$$p_k + \dots, \qquad k = 1 \dots . 4,$$

otherwise would every ordinary point like the origin be translated on a manifoldness by the transformations of our group; that is, the group would be intransitive. Thus in a primitive G_r in R_4 there must always be $(r-4)$ independent infinitesimal transformations of the form

$$\overline{X}_\rho f \equiv \sum_{ik}^{1\dots 4} \beta_{ik\rho} x_i p_k + \sum_{ikj}^{1\dots 4} \gamma_{ikj\rho} x_i x_k p_j + \dots,$$

$$\rho = 1 \dots r-4,$$

and the $\overline{X}_\rho f$ must satisfy relations of the form

$$(\overline{X}_i \overline{X}_k) = \sum_{1}^{r-4} {}_s\, c_{iks} \overline{X}_s f. \qquad (\gamma)$$

The directions through the origin will be transformed by

$$Y_\rho f \equiv \sum_{ik}^{1\dots 4} \beta_{ik\rho} x_i p_k, \qquad \rho = 1 \dots r-4,$$

exactly as by $\overline{X} f_1 \dots \overline{X}_{r-4} f$; and if we substitute for $X_1 f \dots X_{r-4} f$ their values in (γ), we find

$$(Y_i Y_k) \equiv \sum_{1}^{r-4} {}_s\, d_{iks} Y_s f, \qquad i, k = 1 \dots r-4.$$

That is, our primitive G_r in R_4 transforms the directions through the origin when we hold this point just as the initial terms of its transformations IO do. These initial terms of the transformations IO form, taken by themselves, a group of linear homogeneous transformations.

Thus to attack the problem of finding all primitive G_r in R_4 we shall first find all linear homogeneous G_r, $Y_k f$, in R_4, which leave no direction and no pencil of ∞^2 directions through an ordinary point, which we hold, invariant. Then we take the transformations of these linear homogeneous groups as initial terms of the transformations *IO* of the primitive G_r which we seek.

But since the $Y_k f$ are linear and homogeneous in $x_1 \dots x_4$, we may consider the $x_1 \dots x_4$ as homogeneous coordinates in R_3. Thus to find the *linear homogeneous* G_r in R_4, we must write the *projective* groups in R_3 homogeneous and linear in $x_1 \dots x_4$.

We shall then choose those of these linear homogeneous G_r which leave no direction, and no pencil of ∞^2 directions, through an ordinary point which we hold, invariant, and determine the combinations of the primitive G_r and the G_r themselves.

In calculating our primitive G_r we shall make use of an important theorem of Lie. He has proved that *if we can give a* $\overline{G}_r$ *which has the same combination as a required* G_r, *and if the initial terms of the transformations of the two groups are the same, then are the two groups* SIMILAR.

Suppose now we have a G_r in R_4,

$$X_1f \dots . X_rf,$$

where the transformations have been developed in series, so that

$$X_kf \equiv \sum_{1}^{4}{}_{j}\, \xi^{(\lambda)}_{kj}p_j + \dots . ,$$

where we only write the initial terms on the right hand. Then the expressions

$$\overline{X}_kf \equiv \sum_{1}^{1}{}_{j}\, \xi^{(\lambda)}_{kj}p_j$$

form themselves a G_r. If now

$$(\overline{X}_i\overline{X}_k) \equiv \sum_{1}^{r}{}_{s}\, c_{iks}\overline{X}_sf,$$

and at the same time

$$(X_iX_k) = \sum_{1}^{r}{}_{s}\, c_{iks}X_sf$$

where c_{iks} are *the same* constants in both equations, *then we say that* $X_1f \dots . X_rf$ *are connected by normal relations.*

D. Lie has found all projective G_r in R_3 and has classified them according to the figures they leave invariant. He distinguishes

(1). The general projective G_r which leaves no figure invariant.
(2). " " " " " plane "
(3). " " " " " linear complex invariant.
(4). " " " " " straight line "
(5).* " " " " " F_2 "
(6). " " " " " twisted curve *IIIO* invariant.
(7). " " " " " point invariant.

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* "F_2" is sometimes used for "Surface of second degree."

Lie has already found the primitive G_r in R_4 given by cases (1) and (5); we give them at the end of this paper.

We may exclude case (7) at once, since that evidently assigns to each point of general position, which we hold, an invariant direction, so that the groups are imprimitive. Further, it is easy to prove by the theory of Pfaff's Problem, as Lie has shown, that case (2) gives only imprimitive G_r.

Thus we have only the cases (3), (4), and (6) to discuss. The cases (3) and (6) give only one projective G_r each, which assign to a point of general position, which we hold, no invariant direction. But there are a large number of subgroups of the general G_r in case (4) which may give primitive G_r in R_4.

Thus our first object will be to find these subgroups in case (4).

E. The general projective group in R_3 which leaves a straight line which we may choose as $z = \text{const.} = c$, invariant,

has the form

$$\begin{aligned} p,\ q,\ zp,\ zq \ldots\ldots (T_k),\\ xq,\ xp - yq,\ yp \ldots\ldots (U_k),\\ r,\ xp + yq + 2zr,\ xzp + yzq + z^2r \ldots\ldots (V_k),\\ xp + yq \ldots\ldots (W). \end{aligned}$$

We see that the transformations T_k are in pairs commutable, and that they leave each of the planes $z = c$, and each point on the straight line $z = c$, absolutely invariant. Further, the U_k form a group which leaves the planes $z = c$ singly invariant, but transforms the points on the straight line $z = c$. We see that the T_k form an invariant subgroup of the group U_k, T_k.

The V_k form a group, of the same combination as the U_k, which leaves the points on the straight line $z = c$ invariant, but transforms the planes $z = c$. And the T_k also form an invariant subgroup of T_k, V_k. U_k and V_k, by the Principle of Duality, are equally privileged groups.

The transformation W leaves the points on the line $z = c$ as well as the planes $z = c$ both absolutely invariant. We see that T_k, U_k, V_k form a G_{10}—a subgroup of our G_{11}—of which each transformation is absolutely invariant under W.

From what we said above we see that the subgroups which we seek must leave no point or plane in R_3 invariant. Thus:

We wish to find all subgroups of the above G_{11} which leave the straight line $z = c$, but no point or plane, invariant.

Consider two transformations of our G_{11}:

$$\sum^{i}(\alpha_{ki}T_i+\beta_{ki}U_i+\gamma_{ki}V_i+\delta_k W),$$
$$\sum^{j}(\alpha_{ji}T_i+\beta_{ji}U_i+\gamma_{ji}U_i+\delta_j W).$$

If we combine these two we get a transformation in which the new β depend only on the old β, and the new γ on the old γ. No δ occurs in the new transformation. Thus we conclude that

$$\sum^{i}(\beta_{ki}U_i+\gamma_{ki}V_i+\delta_k W)$$

form themselves a group.

Further, we see that

$$\sum^{i}\beta_{ki}U_i,\ \sum^{i}\gamma_{ki}V_i$$

form also a group.

If we compare the group $\sum^{i}\beta_{ki}U_i$ with the original G_{11}, we see that the two groups transform the points on the line $z=c$ in exactly the same manner; and also the group $\sum^{i}\gamma_{ki}V_i$ transforms the planes $z=c$ just as the G_{11} does.

Thus if the group $\sum^{i}\beta_{ki}U_i$ contains less than three members, a point on the straight line $z=c$ will be invariant,* which case is excluded. Same way must the group $\sum^{i}\gamma_{ki}V_i$ contain three members, otherwise we would have an invariant plane.

Thus we wish to find all subgroups of the group

$$\sum^{i}\beta_{ki}U_i,\ \sum^{i}\gamma_{ki}V_i$$

which transform the above-mentioned points and planes so that no point or plane is invariant. Hence, since all the U must occur, we may write these subgroups,

$$U_1+\sum_{1}^{3}{}^{i}\alpha_{1i}V_i,\ U_2+\sum_{1}^{3}{}^{i}\alpha_{2i}V_i,\ U_3+\sum_{1}^{3}{}^{i}\alpha_{3i}V_i,\ \sum^{i}\beta_i V_i.$$

But since all the V must occur, we may write the required groups,

$$V_1+\sum_{1}^{3}{}^{i}\delta_{1i}U_i,\ V_2+\sum_{1}^{3}{}^{i}\delta_{2i}U_i,\ V_3+\sum_{1}^{3}{}^{i}\delta_{3i}U_i,\ \sum^{i}\gamma_i U_i.$$

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* By a theorem of Lie's on the groups of the M_1.

Since U and V are equally privileged, we easily see that the only possibilities here are the two groups

$$V_1 + U_1, \; V_2 - U_2, \; V_3 - U_3, \text{ and } V_1, \; V_2, \; V_3, \; U_1, \; U_2, \; U_3.$$

We shall call these two our "abridged" (verkürzte) groups. The G_r which we seek must all contain the transformations of one of these abridged groups. Since the T_k and W transform neither the points nor the planes mentioned above, these transformations may occur in the required G_r free, or added with constant coefficients to the transformations of one of the abridged groups.

(A). Let us take up the case where the transformations of the first of the above abridged groups occur in our sought subgroups; that is, the transformations of $V_1 + U_1, \; V_2 - U_2, \; V_3 - U_3$ occur.

(1). Suppose for the present no transformation W occurs at all; and also no free transformation of the form $\sum_1^4 {}^{\mu} \delta_\mu T_\mu$. Then the transformations of our sought groups can only have the forms

$$V_1 + U_1 + \sum{}^{i} \alpha_i T_i, \; V_2 - U_2 + \sum{}^{j} \beta_j T_j, \; V_3 - U_3 + \sum{}^{k} \gamma_k T_k.$$

Let us introduce new variables into these transformations by means of the substitution

$$x' = x - \alpha_1 z + \alpha_2, \quad y' = y + \frac{\beta_2}{2}.$$

Since nothing depends upon the symbols we use, we can drop the accents from the new transformations, and our substitution is equivalent to putting

$$\alpha_1 = \alpha_2 = \beta_2 = 0.$$

We now easily find by combining our transformations, and remembering that they must form a group, that the only possible group is

$$r + xq, \; 2(yq + zr), \; xzp + yzq + z^2 r - yp.$$

It is easy to see that this projective group leaves the F_2,

$$y - zx = 0$$

and all generators of one family, absolutely invariant.

If the transformation W occurs, it is easy to see that it cannot occur additively, *i. e.* only free. Hence we get the group

$$r + xq, \; xp + yq, \; 2(yq + zr), \; xzp + yzq + z^2 r - yp.$$

This group leaves the F_2,

$$y - zx = 0$$

and two different generators of one family invariant.

(2). Suppose a transformation of the form

$$\sum_{1}^{4}{}^{\mu}\,\delta_\mu T_\mu$$

occurs free. Suppose for the present that no W occurs; our transformations are

$$r + xq + \sum{}^{i}\,\alpha_i T_i, \quad 2(yq + zr) + \sum{}^{j}\,\beta_j T_j, \quad (xz - y)p + yzq + z^2 r + \sum{}^{k}\,\gamma_k T_k$$

$$\sum \delta_\mu T_\mu \equiv \delta_1 T_1 + \delta_2 T_2 + \delta_3 T_3 + \delta_4 T_4.$$

By making the same change of variables as in (1), we can again put

$$\alpha_1 = \alpha_2 = \beta_2 = 0.$$

By combination we find that the transformations

$$(\delta_2 + \delta_3)(p - zq) + (\delta_4 - \delta_1)(q + zp),$$
$$\delta_2 q - \delta_3 zp$$

must occur.

(I). Suppose $\delta_2 = -\delta_3$ and

(a). $$\delta_3 = 0.$$

Then we see $$(\delta_4 - \delta_1)(q + zp)$$

must occur.

(i). Suppose $\delta_1 = \delta_4$. Then must $\delta_1 \times \delta_4 \neq 0$. Hence

$$p + zq$$

occurs. Any other transformation $\sum{}^{i}\,\alpha_i T_i$, which occurs has the form

$$ap + bq + czp.$$

Combine this transformation with $r + xq + \sum{}^{i}\,\alpha_i T_i$, and we see

$$c(p - zq) - aq,$$

and $$2cq$$

occur. If here $c \neq 0$ we see

$$p, q, zp, zq,$$

must all occur. Thus we find the group

$$r+xq,\ yq+zr,\ (xz-y)p+yzq+z^2r,\ p,\ zp,\ q,\ zq.$$

If the transformation W occurs, we see that it can only occur free, and we find a group of which the last is a subgroup,

$$r+xq,\ yq+zr,\ (xz-y)p+yzq+z^2r,\ xp+yq,\ p,\ zp,\ q,\ zq.$$

It is easy to see that these groups leave two coinciding straight lines invariant.

If above we have $c=0$ and $a \neq 0$, we find these groups again. Thus

$$c=a=0,$$

and we see then

$$b=0.$$

Our transformations now have the forms

$$r+xq+\alpha_3 zp+\alpha_4 zq,\ 2(yq+zr)+\beta_3 zp+\beta_4 zq,$$

$$(xz-y)p+yzq+z^2r+\sum_2^4{}^i\gamma_i T_i,\ p+zq.$$

By combination we easily find

$$\gamma_i=\alpha_3=\beta_4=0,\ \beta_3=-\alpha_4.$$

If now $\alpha_4=0$ we find the group

$$r+xq,\ yq+zr,\ (xz-y)p+yzq+z^2r,\ p+zq.$$

If $\alpha_4 \neq 0$, introduce new variables by means of the substitution

$$x'=\frac{x}{\alpha_4}+\frac{z}{2},\ y'=\frac{y}{\alpha_4},$$

and we find our last group again.

This group leaves the surface of second degree

$$y-zx=0$$

and two coinciding generators of the same invariant.

If the transformation $xp+yp$ occur, it can only occur free, which gives the group

$$r+xq,\ yq+zr,\ (xz-y)p+yzq+z^2r,\ p+zq,\ xp+yq.$$

This group leaves a surface second degree invariant, and also one of its generators.

(ii). Suppose now $\delta_1 \neq \delta_4$.

Thus we see $q + zp$ occurs. Combine this transformation with $r + xq + \sum^{i} \alpha_i T_i$ and we see

$$p - zq,\ q$$

occur also.

We see at once that no other transformation $\sum^{\mu} \delta_\mu T_\mu$ can occur, unless *all* occur, a case which we have already finished.

We easily find the group

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2 r,\ q,\ zp,\ p - zq.$$

The transformation $xp + yq$ may occur free, which gives a group of which the last is a subgroup,

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2 r,\ xp + yq,\ q,\ zp,\ p - zq.$$

Lie has shown that these groups leave a linear complex and one of its straight lines invariant.

We now come to the case

(b). $$\delta_2 = - \delta_3,\ \delta_2 . \delta_3 \neq 0.$$

We see at once that the transformation $q + zp$ occurs. Combine this with $r + xq + \sum^{i} \alpha_i T_i$ and then we find that also

$$p - zq,\ q$$

must occur. Thus we have the same groups as in the last case.

(II). $$\delta_2 \neq - \delta_3.$$

Thus, $$p - zq + k(q + zp), \quad k = \frac{\delta_4 - \delta_1}{\delta_2 + \delta_3}$$

occurs; also $$\delta_2 q - \delta_3 zp.$$

(a). If now $\delta_3 \neq 0$, then $zp - \dfrac{\delta_2}{\delta_3} q$ occurs. Combine with

$$r + xq + \sum^{i} \alpha_i T_i$$

and we see $$p - zq,\ q$$

occur. Thus we have again the same case as above.

(b). If $\delta_3 = 0$, then $\delta_2 \neq 0$ and q must occur, with $p - zq + k . zp$. If $k \neq 0$, we see that we get the above case again.

If $k = 0$, we have

$$q,\ p - zq.$$

Combine the last transformation with

$$(xz - y)p + yzq + z^2 r + \sum^{i} \gamma_i T_i$$

and we see that we get still our often-mentioned case above.

These are all the groups possible in category (A).

(B). Let us now take up the case where the transformations of the abridged group

$$U_1,\ U_2,\ U_3;\ V_1,\ V_2,\ V_3$$

occur in our sought groups.

(1). Suppose no transformation $\Sigma\delta T$ occur free. Then we have

$$U_1+\sum{}^i \alpha_i T_i,\ U_2+\sum{}^j \beta_j T_j,\ U_3+\sum{}^k \gamma_k T_k,$$

and

$$V_1+\sum{}^i a_i T_i,\ V_2+\sum{}^j b_j T_j,\ V_3+\sum{}^k c_k T_k.$$

We easily find that this gives the group

$$xq,\ xp-yq,\ yp;\ r,\ xp+yq+2zr,\ (xz-y)p+yzq+z^2r.$$

The transformation $xp+yq$ may occur free, which gives the group

$$xq,\ xp,\ yq,\ yp;\ r,\ xp+yq+2zr,\ (xz-y)p+yzq+z^2r.$$

These groups leave two different straight lines in R_3 invariant.

(2). If one transformation $\sum{}^\mu \delta_\mu T_\mu$ occurs free, it is easy to see that *all* the T must then occur free. Thus we find the group

$$xq,\ xp-yq,\ yp;\ r,\ xp+yq+2zr,\ xzp+yzq+z^2r-yp,\quad p,\ q,\ zp,\ zq.$$

As usual, $xp+yq$ may occur free; and we find so the original G_{11}, of which the last is a subgroup which leaves a straight line invariant.

These are *all* of our required groups.

Lie has given general formulae for writing projective transformations in $(n-1)$ variables linear and homogeneous in n variables. For writing projective transformations in xyz linear and homogeneous in $x_1x_2x_3x_4$, the formulae are

$$\begin{aligned}
&xq\equiv x_1p_2,\ xr\equiv x_1p_3,\ yp\equiv x_2p_1,\ yr\equiv x_2p_3,\\
&zp\equiv x_3p_1,\ zq\equiv x_3p_2,\\
&p\equiv x_4p_1,\quad q\equiv x_4p_2,\quad r\equiv x_4p_3,\\
&x.U\equiv -x_1p_4,\ y.U\equiv -x_2p_4,\ z.U\equiv -x_3p_4,\\
&U+xp\equiv x_1p_1-x_4p_4,\ U+yq\equiv x_2p_2-x_4p_4,\\
&U+zr\equiv x_3p_3-x_4p_4,\\
&xp\equiv x_1p_1-\frac{1}{4}\sum_1^4{}^i x_ip_i,\ yq\equiv x_2p_2-\frac{1}{4}\sum_1^4{}^i x_ip_i,\\
&zr\equiv x_3p_3-\sum_1^4{}^i x_ip_i,
\end{aligned}$$

where we write for brevity

$$U\equiv xp+yq+zr\,.$$

If we reverse these formulae of transformation, and write the transformation

$$\overline{U}\equiv\sum_{1}^{4}{}^{i}\,x_ip_i$$

in the variables xyz, we get *zero*, *i. e.* the "*identical*" transformation. Thus we see that when we write our projective G_r, in xyz, linear and homogeneous in $x_1x_2x_3x_4$, the transformation $\overline{U}$ may also occur in the linear homogeneous group, free, or added with a constant coefficient to the other transformations.

CHAPTER I.

A Twisted Curve IIIO is Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$p+2xq+3yr,\ xp+2yq+3zr,\ 3x^2p+3xyq+3xzr-2yp-zq\,,$$

and leaves the curve *IIIO*

$$y-x^2=0,\ z-x^3=0\,,$$

invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, the group has the form

$$x_4p_1+2x_1p_2+3x_2p_3,\ x_1p_1-x_2p_2+3\,(x_4p_4-x_3p_3),\ 2x_2p_1+x_3p_2+3x_1p_4\,,$$

to which may be added

$$U\equiv\sum_{1}^{4}{}^{i}\,x_ip_i\,.$$

In the following we shall only write the initial terms of our transformations, for the sake of brevity.

When we have found the transformations of a group in *finite* form we shall enclose them in a frame to catch the eye.

§2.

Thus the transformations IO of the groups we now seek have the forms—writing only the initial terms—

$$\begin{aligned} S_1 &\equiv x_4p_1 + 2x_1p_2 + 3x_2p_3, \\ S_2 &\equiv x_1p_1 - x_2p_2 + 3(x_4p_4 - x_3p_3), \\ S_3 &\equiv 2x_2p_1 + x_3p_2 + 3x_1p_4, \end{aligned}$$

to which may be added

$$U \equiv \sum_1^4{}^i x_ip_i.$$

For the initial terms of these transformations the relations hold

$$(S_1S_2) = -2S_1,\ (S_1S_3) = S_2,\ (S_2S_3) = -2S_3,\ (S_iU) = 0.$$

Thus we see that if the transformation U does not occur free, it cannot occur at all.

We wish to see what transformations of an order higher than the first can occur.

If a transformation IO occurs it must have the form

$$Xf \equiv \xi_1(x_1x_2x_4)p_1 + \xi_2(x_1x_2x_3)p_2 + \xi_3(x_2x_3)p_3 + \xi_4(x_1x_4)p_4.$$

For we know that the four transformations zero $O:p_k$ must occur; thus, if ξ_1 were a function of x_3, we could by a proper combination of Xf with the p_k get a transformation IO of the form

$$Yf \equiv x_3p_1 + \eta_2p_2 + \eta_3p_3 + \eta_4p_4.$$

But Yf must be a transformation of our group, and so, linear in the S_i. We see at once that this is impossible, hence ξ_1 is free of x_3. Same way we see that ξ_2, ξ_3, ξ_4 have the above forms.

Since the ξ_k are analytical functions of their arguments, we may write

$$\begin{aligned} Xf \equiv\ & (a_1x_1^2 + b_1x_2^2 + c_1x_4^2 + d_1x_1x_2 + e_1x_1x_4 + f_1x_2x_4)p_1 + (a_3x_2^2 + b_3x_2x_3 + c_3x_3^2)p_3 \\ & + (a_2x_1^2 + b_2x_2^2 + c_2x_3^2 + d_2x_1x_2 + e_2x_1x_3 + f_2x_2x_3)p_2 + (a_4x_1^2 + b_4x_1x_4 + c_4x_4^2)p_4 \end{aligned}$$

where a, b, ... c_4 are certain constants.

Combine now Xf with p_3, and we get a transformation IO,

$$Z_1f \equiv (2c_2x_3 + e_2x_1 + f_2x_2)p_2 + (b_3x_2 + 2c_3x_3)p_3,$$

which must belong to our group. Thus we must have

$$Z_1f \equiv \sum_1^3{}^k \alpha_kS_k + \alpha U,$$

where the α are certain constants. But we see at once that this relation gives

$$c_2 = e_2 = f_2 = b_3 = 2c_3 = 0.$$

Hence

$$Xf \equiv (a_1x_1^2 + b_1x_2^2 + c_1x_4^2 + d_1x_1x_2 + e_1x_1x_4 + f_1x_2x_4)\,p_1 + a_3x_2^2\,p_3 \\ + (a_2x_1^2 + b_2x_2^2 + d_2x_1x_2)\,p_2 + (a_4x_1^2 + b_4x_1x_4 + c_4x_4^2)\,p_4.$$

Now combine Xf with p_4, and we find a transformation IO,

$$Z_2f \equiv (2c_1x_4 + e_1x_1 + f_1x_2)\,p_1 + (b_4x_1 + 2c_4x_4)\,p_4,$$

which must belong to our group. Thus as before we must have

$$Z_2f \equiv \sum_1^3{}^i\, \beta_iS_i + \beta U,$$

where the β are certain constants. We find without difficulty that this gives

$$c_1 = e_1 = f_1 = b_4 = c_4 = 0.$$

Proceeding in exactly the same manner with Xf and p_1 and p_2 we find

$$a_1 = d_1 = a_2 = d_2 = a_4 = b_1 = b_2 = a_3 = 0.$$

Thus we find that all constants are zero; that is, that no transformation IIO can occur at all. Also none of $IIIO$ can occur, for a transformation $IIIO$ combined with a transformation zero O gives a transformation IIO, which cannot occur. We see in this manner that *no transformation of an order higher than the first can occur.*

We have now two cases according as the transformation U occurs or not.

§3.

Suppose the transformation U occurs. In this case the transformations of our group are

$$p_1,\, p_2,\, p_3,\, p_4;\;\; S_1 \equiv x_4p_1 + 2x_1p_2 + 3x_2p_3,\;\; U \equiv \sum_1^4{}^i\, x_ip_i,$$

$$S_2 \equiv x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3)\,;$$

$$S_3 \equiv 2x_2p_1 + x_3p_2 + 3x_1p_4.$$

Let us find the composition of our group. We see that the S_i and U are connected by normal relations, and so are the S_i among each other. We shall first choose the transformations zero O so that they are connected with U by normal relations, *i. e.* we shall "normalize" with U, and then we shall see how the transformations zero O are connected with the other transformations IO and with each other.

If now we combine p_1 and U we must get a transformation of our group, thus,

$$(p_1U) = p_1 + \sum_1^3{}^i \alpha_i S_i + \beta U, \quad (\alpha_i, \beta \text{ consts.}).$$

Let us introduce a new p_1 by means of the substitution

$$\bar{p}_1 \equiv p_1 + \sum_1^3{}^i A_i S_i + BU,$$

where A_i and B are certain constants. Then we have

$$(\bar{p}_1 U) \equiv \bar{p}_1 - \sum_1^3{}^i A_i S_i - BU + \sum_1^3{}^i \alpha_i S_i + \beta U.$$

Now put
$$A_i = \alpha_i, \quad B = \beta,$$
and we have
$$(\bar{p}_1 U) = \bar{p}_1.$$
Since nothing depends upon the symbol we use, we may write this

$$(p_1 U) = p_1,$$

remembering that this is a new p_1.

Proceeding in just the same manner we may write

$$\begin{cases} (p_2 U) = p_2, \\ (p_3 U) = p_3, \\ (p_4 U) = p_4. \end{cases}$$

Thus we have normalized the transformations zero O with U. Let us find the relations between the p_k and the S_i. We have

$$(p_1 S_1) = 2p_2 + \sum_1^3{}^i a_i S_i + bU.$$

Form Jacobi's Identity with p_1, S_1 and U, thus

$$((p_1S_1)\,U) + ((S_1U)\,p_1) + ((Up_1)\,S_1) = 0,$$
or
$$2\,(p_2U) - (p_1S_1) = 0,$$
and
$$(p_1S_1) = 2p_2.$$

Thus p_1 and S_1 are connected by a normal relation. In exactly the same manner we find that the other transformations zero O and IO are connected by normal relations.

It remains to see how the transformations zero O are connected among each other.

We cannot say *a priori* of what order a transformation will be which we obtain by combining two transformations of zero O, since no transformations of *minus first order* can occur.

Thus we must write

$$(p_1p_2)=\sum_1^4{}^i a_ip_i+\sum_1^3{}^k b_kS_k+cU,$$

where a_i, b_k, c are certain constants.

Now form Jacobi's Identity with p_1, p_2 and U, thus

$$((p_1p_2)\,U)+((p_2U)\,p_1)+((Up_1)\,p_2)=0,$$

or

$$\sum_1^4{}^i a_ip_i-2\,(p_1p_2)=0,$$

i. e.

$$(p_1p_2)=0.$$

In the same way we find

$$(p_ip_k)=0.$$

Thus we have found the composition of our group, and find that all transformations are connected by normal relations.

A group of this composition whose transformations have the same initial terms is

$$\boxed{\begin{gathered}p_1,\,p_2,\,p_3,\,p_4;\;x_1p_1-x_2p_2+3\,(x_4p_4-x_3p_3)\,;\;2x_2p_1+x_3p_2+3x_1p_4,\\ x_4p_1+2x_1p_2+3x_2p_3,\;x_1p_1+x_2p_2+x_3p_3+x_4p_4.\end{gathered}}$$

By Lie's theorem, p. 303, this is thus the group we seek. This group is *primitive*, for with any ordinary point which we hold is, as is easy to see, no plane pencil of directions invariantly connected, and also no direction.

§4.

Suppose U does not occur.

Here we have the four transformations zero O and the transformations IO,

$$\begin{aligned}S_1&=x_4p_1+2x_1p_2+3x_2p_3,\\ S_2&=x_1p_1-x_2p_2+3\,(x_4p_4-x_3p_3),\\ S_3&=2x_2p_1+x_3p_2+3x_1p_4.\end{aligned}$$

We see that the transformations IO are connected by normal relations. To find the composition of our group, let us normalize with S_2. We have

$$(p_1S_2)=p_1+\sum_1^3{}^i\alpha_iS_i, \quad (\alpha_i \text{ const.}),$$

and when we introduce a new p_1 by means of

$$\bar{p}_1\equiv p_1+\sum_1^3{}^i A_iS_i, \quad (A_i \text{ const.}),$$

we get

$$(\bar{p}_1S_2)=\bar{p}_1-\sum_1^3{}^i A_iS_i+\sum_1^3{}^i\alpha_iS_i-2A_1S_1+2A_3S_3.$$

We can evidently choose our constants A_i so that

$$(\bar{p}_1S_2)=\bar{p}_1,$$

or, as we may write it, $$(p_1S_2)=p_1.$$

In the same way we find

$$\begin{cases}(p_2S_2)=-p_2,\\(p_3S_2)=-3p_3,\\(p_4S_2)=3p_4.\end{cases}$$

Now we wish to see how the p_k are connected with S_1 and S_3. We have

$$(p_1S_1)=2p_2+\sum_1^3{}^i a_iS_i, \quad (a_i \text{ const.}).$$

Form Jacobi's Identity with p_1, S_1 and S_2, thus

$$((p_1S_1)S_2)+((S_1S_2)p_1)+((S_2p_1)S_1)=0,$$

or $$(p_1S_1)=2p_2.$$

Thus we see p_1 and S_1 are connected by a normal relation, and in same manner we find that all (p_kS_i) are normal.

It only remains to see how the transformations zero O are connected among each other. We have

$$(p_1p_2)=\sum_1^4{}^i\alpha_ip_i+\sum_1^3{}^k\beta_kS_k, \quad (\alpha, \beta \text{ const.}).$$

Form Jacobi's Identity with p_1, p_2 and S_2; thus

$$((p_1p_2)S_2)-(p_1p_2)+(p_1p_2)=0,$$

or $$\alpha_1=\alpha_2=\alpha_3=\alpha_4=\beta_1=\beta_3=0.$$

Thus, $$(p_1p_2) = \beta_2 S_2.$$

Same way $$\begin{cases} (p_1p_3) = \gamma_1 S_1, \\ (p_1p_4) = (p_2p_3) = 0, \\ (p_2p_4) = \delta_3 S_3, \\ (p_3p_4) = \varepsilon_2 S_2. \end{cases}$$

Form Jacobi's Identity with p_1, p_2 and p_3; thus

$$((p_1p_2)p_3) + ((p_2p_3)p_1) + ((p_3p_1)p_2) = 0,$$

or $$3\beta_2 p_3 + 3\gamma_1 p_3 = 0, \quad \gamma_1 = -\beta_2.$$

Also $$((p_1p_2)\, S_1) - 3\,(p_1p_3) = 0,$$

or $$2\beta_2 S_1 - 3\gamma_1 S_1 = 0, \quad i.\,e.\ \beta_2 = \gamma_1 = 0.$$

Similarly we find without difficulty by means of Jacobi's Identity

$$\delta_3 = \varepsilon_2 = 0.$$

Hence all $$(p_ip_k) = 0,$$

and our transformations are connected by normal relations. Our group is evidently

$$\boxed{\begin{array}{c} p_1,\, p_2,\, p_3,\, p_4;\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4; \\ x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3). \end{array}}$$

This group is a subgroup of the one found in §3, and we see at once that this one is also *primitive* for the same reason that that one is.

CHAPTER II.

A Linear Complex is Invariant in R_3.

§1.

The projective group written in xyz has the form

$$p - yr,\ q + xr,\ r,\ xq,\ yp,\ xp - yq,\ xp + yq + 2zr,$$
$$zq + x.U,\ zp - y.U,\ xzp + yzq + z^2r,\ (\text{where } U \equiv xp + yq + zr),$$

and leaves the linear complex

$$dz + xdy - ydx = 0$$

invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, the group has the form

$$x_4p_1-x_2p_3,\ x_4p_2+x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1-x_2p_2,\ x_2p_1,$$
$$x_3p_1+x_2p_4,\ x_3p_3-x_4p_4,\ x_3p_2-x_1p_4,\ x_3p_4,$$

to which may be added

$$U\equiv\sum_1^4{}^i\,x_ip_i.$$

§2.

Thus the transformations *IO* of the groups we now seek have the forms

$$S_1\equiv x_4p_1-x_2p_3,\ S_2\equiv x_4p_2+x_1p_3,\ S_3\equiv x_4p_3,\ S_4\equiv x_1p_2,$$
$$S_5\equiv x_1p_1-x_2p_2,\ S_6\equiv x_2p_1,\ S_7\equiv x_3p_1+x_2p_4,\ S_8\equiv x_3p_3-x_4p_4,$$
$$S_9\equiv x_3p_2-x_1p_4,\ S_{10}\equiv x_3p_4,$$

to which may be added

$$U\equiv\sum_1^4{}^i\,x_ip_i.$$

Let us see what transformations of an order higher than the first can occur.

Let s be the maximum order, so that we suppose a transformation of form

$$Xf\equiv\xi_1^{(s)}p_1+\xi_2^{(s)}p_2+\xi_3^{(s)}p_3+\xi_4^{(s)}p_4$$

occurs, where the index (s) shows that the ξ_k are of the s^{th} degree.

Since in our group x_1, x_2 are equally privileged with x_3, x_4, we can suppose that $\xi_1^{(s)}$ and $\xi_2^{(s)}$ are not both zero. If now $\xi_1^{(s)}\neq 0$, by combining Xf with

$$x_1p_2,\ x_1p_3-x_4p_2,\ x_1p_4-x_3p_2$$

in proper order, we see that

$$Yf\equiv x_1^sp_1+\bar{\xi}_2^{(s)}p_2+\bar{\xi}_3^{(s)}p_3+\bar{\xi}_4^{(s)}p_4$$

occurs. Combine p_1 and Yf, hence

$$Zf\equiv sx_1^{s-1}p_1+y_2^{(s-1)}p_2+y_3^{(s-1)}p_3+y_4^{(s-1)}p_4$$

must be a transformation of our group.

Combine now Zf and Yf, hence

$$-sx_1^{2s-2}p_1+\xi_2^{(2s-2)}p_2+\xi_3^{(2s-2)}p_3+\xi_4^{2s-2}p_4$$

must occur. But s is the maximum order, hence

$$2s-2<s+1,\ \text{or}\ s<3.$$

Thus when $\xi_1^{(s)}\neq 0$, $s<3$.

If $\xi_1^{(s)} = 0$, then $\xi_2^{(s)} \neq 0$, and

$$Xf \equiv \xi_2^{(s)} p_2 + \xi_3^{(s)} p_3 + \xi_4^{(s)} p_4 .$$

Combine this transformation with

$$x_2 p_1, \ x_2 p_3 - x_4 p_1, \ x_2 p_4 + x_3 p_1$$

in proper order, and we find that

$$x_2^s p_2 + \overline{\overline{\xi}}{}_1^{(s)} p_1 + \overline{\overline{\xi}}{}_3^{(s)} p_3 + \overline{\overline{\xi}}{}_4^{(s)} p_4$$

must occur, and proceeding just as with Yf above, we see $s < 3$.

Thus at all events is $s < 3$.

Now let us see what transformations IIO, if any, can occur. If any transformation IIO occurs, we wish now to show that one of the form

$$\overline{\xi}_2^{(2)} p_2 + \overline{\xi}_3^{(2)} p_3 + \overline{\xi}_4^{(2)} p_4$$

must occur, where $\overline{\xi}_2^{(2)} \neq 0$.

First we must prove that no transformation IIO of the form

$$Xf \equiv \xi_1 p_1 + \xi_2 p_2$$

can occur, and since x_1, x_2 are equally privileged with x_3, x_4, of course then none of the form

$$\xi_3 p_3 + \xi_4 p_4$$

could occur.

If Xf occurs, then must ξ_1 be $\neq 0$. For if not, a transformation

$$\xi_2 p_2 \equiv x_1^2 p_2$$

must occur. Combine this transformation with $x_2 p_1$ and we see

$$2 x_1 x_2 p_2 - x_1^2 p_1$$

must occur. Combine the last transformation with $x_1^2 p_2$ and we get a transformation $IIIO$, which is impossible.

Thus $\xi_1 \neq 0$ when Xf occurs.

By combining Xf properly with $x_3 p_4$ and $x_1 p_2$, we can arrange so that

$$\overline{\xi}_1 (x_1 x_3) p_1 + \overline{\xi}_2 p_2$$

must occur. But from the transformations IO we see that ξ_1 can contain no x_3 here. Thus a transformation

$$\overline{X}f \equiv x_1^2 p_1 + \overline{\xi}_2 p_2$$

occurs. Combine $x_1 p_1 - x_2 p_2$ with $\overline{X}f$ and we see

$$Yf \equiv x_1^2 p_1 + \left\{ x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2} + \xi_2 \right\} p_2$$

must be a transformation of our group. Here Yf must be the same transformation as $\overline{X}f$, otherwise, by subtracting, we would find a transformation

$$\eta_2 p_2$$

which would belong to our group. But we saw no such transformation can occur; thus

$$Yf \equiv \overline{X}f,$$

and hence

$$x_1 \frac{\partial \xi_2}{\partial x_1} + x_2 \frac{\partial \xi_2}{\partial x_2} = 0, \text{ or } \xi_2 = \xi_2(x_1 x_2,\, x_3 x_4).$$

Hence

$$\overline{X}f \equiv x_1^2 p_1 + (a x_1 x_2 + b x_3^2 + c x_3 x_4 + d x_4^2)\, p_2.$$

Combine this transformation with p_3, p_4 and we see

$$b = c = d = 0;$$

thus

$$x_1^2 p_1 + a x_1 x_2 p_2$$

occurs. Combine this with p_1 and we see at once

$$a = -2.$$

Combine now

$$x_1^2 p_1 - 2 x_1 x_2 p_2$$

with $x_1 p_2$ and we find that

$$x_1^2 p_2$$

must occur. But we saw that this was impossible; thus, no transformation of form

$$\xi_1^{(2)} p_1 + \xi_2^{(2)} p_2 \text{ or } \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

can occur. Thus if a transformation of form

$$\xi_1^{(2)} p_1 + \xi_2^{(2)} p_2 + \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

occurs, but not

$$Zf \equiv \eta_2^{(2)} p_2 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4,$$

we must suppose that $\xi_1^{(2)} \neq 0$. Hence

$$Xf \equiv x_1^2 p_1 + \xi_2^{(2)} p_2 + \xi_3^{(2)} p_3 + \xi_4^{(2)} p_4$$

must occur. Combine this transformation with $x_1 p_1 - x_2 p_2$ and we see

$$Yf \equiv x_1^2 p_1 + \left(x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2} + \xi_2\right) p_2 + \left(x_1 \frac{\partial \xi_3}{\partial x_1} - x_2 \frac{\partial \xi_3}{\partial x_2}\right) p_3 + (\;)\, p_4$$

must belong to our group. Subtract Xf from Yf and we see that

$$\left(x_1 \frac{\partial \xi_2}{\partial x_1} - x_2 \frac{\partial \xi_2}{\partial x_2}\right) p_2 + \left(x_1 \frac{\partial \xi_3}{\partial x_1} - x_2 \frac{\partial \xi_3}{\partial x_2} - \xi_3\right) p_3 + (\;)\, p_4$$

also belongs to the group. But this is a transformation of the form Zf, and since by hypothesis none such can occur, we must have

$$x_1\frac{\partial\xi_2}{\partial x_1} - x_2\frac{\partial\xi_2}{\partial x_2} = 0, \quad x_1\frac{\partial\xi_3}{\partial x_1} - x_2\frac{\partial\xi_3}{\partial x_2} = \xi_3, \quad x_1\frac{\partial\xi_4}{\partial x_1} - x_2\frac{\partial\xi_4}{\partial x_2} = \xi_4,$$

and hence $\xi_2 = \xi_2(x_1x_2, x_3, x_4)$, $\xi_3 = \xi_3(x_1x_2, x_3, x_4).x_1$
and $\xi_4 = \xi_4(x_1x_2, x_3, x_4).x_1$.

Since our transformation is of the *IIO*,

$$\xi_3 = x_1.\xi_3(x_3, x_4), \quad \xi_4 = x_1.\xi_4(x_3, x_4).$$

Thus a transformation of the form

$$x_1^2p_1 + (ax_1x_2 + bx_3x_4 + cx_3^2 + dx_4^2)p_2 + x_1(ex_3 + gx_4)p_3 + x_1(e'x_3 + g'x_4)p_4$$

occurs. By combining this transformation p_1, p_2, p_3, p_4 we easily see that also

$$x_1^2p_1 - 2x_1x_2p_2 + (ex_3x_4 + cx_3^2 + dx_4^2)p_2 + x_1(ex_3 + 2dx_4)p_3 + x_1(2cx_3 + ex_4)p_4$$

must belong to our group. Combine this transformation with x_1p_2 and we see that $x_1^2p_2$ must occur, which we saw to be impossible.

Thus a transformation of the form

$$Zf = \xi_2^{(2)}p_2 + \xi_3^{(2)}p_3 + \xi_4^{(2)}p_4$$

must occur, if any transformation *IIO* occur at all; and here $\xi_2^{(2)} \neq 0$, and not $\xi_3^{(2)}$, $\xi_4^{(2)}$ both can be zero.

Combine properly with x_1p_2 and x_3p_4, and we see that a transformation

$$Xf \equiv \xi_2(x_1, x_3)p_2 + \xi_3p_3 + \xi_4p_4$$

occurs, where all ξ_k are free of x_2. Combine Xf again with x_3p_4 and we find

$$x_3\left(\frac{\partial\xi_3}{\partial x_4}p_3 + \frac{\partial\xi_4}{\partial x_4}.p_4\right) - \xi_3p_4.$$

But no such transformation can occur, hence

$$\xi_3 = \xi_3(x_1x_3), \quad \xi_4 = \frac{x_4}{x_3}.\xi_3 + \eta_4(x_1x_3).$$

From the transformation *IO* we see that no x_1 can occur in ξ_3 here; thus

$$\xi_3 = ax_3^2, \quad \xi_4 = ax_3x_4 + \eta_4(x_1x_3), \qquad (a = \text{const.}).$$

Thus a transformation *IIO* of form

$$\xi_2(x_1x_3)p_2 + ax_3^2p_3 + \{ax_3x_4 + \eta_4(x_1x_3)\}p_4$$

must belong to our group. Combine this with p_3 and we see $a = 0$; thus

$$Xf = \xi_2(x_1x_3)p_2 + \xi_4(x_1x_3)p_4,$$

where $\xi_2 . \xi_4 \neq 0$. By combining Xf properly with $x_1p_3 - x_4p_2$ we see that

$$x_1^2p_4 + \bar{\xi}_2(x_1x_3)p_2$$

occurs, or
$$x_1^2p_4 + (dx_1^2 + bx_1x_3 + cx_3^2)p_2 .$$

Combine with p_1 and p_3 and we see

$$b = -2, \quad c = 0;$$

thus
$$x_1^2p_4 + (dx_1^2 - 2x_1x_3)p_2$$

occurs. Combine with x_4p_3 and we see

$$X_1f = x_1^2p_3 + 2x_1x_4p_2$$

occurs. Combine x_2p_1 with X_1f and we see

$$X_2f \equiv x_1x_4p_1 - x_2x_4p_2 - x_1x_2p_3$$

occurs. Combine X_1f and X_2f and we get a transformation $IIIO$; but such a transformation cannot occur; *thus no transformation IIO can occur.*

We now have two cases according as U occurs or not. We easily see that U cannot occur additively.

§3.

Suppose U occurs free.

In this case we "normalize" with U just as in §3, Chap. I, and find without any difficulty the group

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4; \ x_4p_1 - x_2p_3, \ x_4p_2 + x_1p_3, \ x_4p_3, \ x_1p_1 - x_2p_2, \ x_2p_1, \ x_1p_2, \\ x_3p_1 + x_2p_4, \ x_3p_2 - x_1p_4, \ x_3p_4, \ x_3p_3 - x_4p_4, \ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{array}}$$

We see at once that this group is *primitive*, for the same reason as those found in the last chapter are primitive.

§4.

Suppose U does not occur free.

Our transformations IO are

$$x_4p_1 - x_2p_3, \ x_4p_2 + x_1p_3, \ x_4p_3, \ x_1p_2, \ x_1p_1 - x_2p_2, \ x_2p_1,$$
$$x_3p_1 + x_2p_4, \ x_3p_2 - x_1p_4, \ x_3p_4, \ x_3p_3 - x_4p_4.$$

Let us designate the first nine of these transformations in the order written respectively by $S_1 \dots S_9$, and put

$$T = x_1p_1 - x_2p_2 + (x_3p_3 - x_4p_4).$$

These transformations IO are connected by normal relations, and we see by combination that U cannot occur additively. We may at once choose the p_k, without loss of generality, so that

$$\begin{cases}(p_1T) = p_1, \\ (p_2T) = -p_2, \\ (p_3T) = p_3, \\ (p_4T) = -p_4.\end{cases}$$

Further, we find without difficulty by means of Jacobi's Identity that now all the relations (p_kS_i) are normal. It remains to see how the transformations zero O are connected among each other. We have

$$(p_1p_2) = \sum_1^4{}^i\, b_ip_i + \sum_1^9{}^k\, a_kS_k + bT.$$

Form Jacobi's Identity with p_1, p_2 and T, thus

$$((p_1p_2)\,T) = 0,$$

or $$(p_1p_2) = a_2S_2 + a_5S_5 + a_7S_7 + bT.$$

Also now $$((p_1p_2)\,S_3) = 0, \quad i.\,e.\ a_7 = b = 0.$$

Finally $$((p_1p_2)\,S_6) = 0, \quad i.\,e.\ a_2 = a_5 = 0.$$

Thus $$(p_1p_2) = 0.$$

Analogously we find $$(p_ip_k) = 0.$$

Thus we find that all of our transformations are connected by normal relations. Our group is evidently

$$\boxed{\begin{gathered}p_1,\ p_2,\ p_3,\ p_4;\ x_1p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1 - x_2p_2; \\ x_2p_1,\ x_3p_1 + x_2p_4,\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_3p_3 - x_4p_4.\end{gathered}}$$

This is a subgroup of the one in §3, and is, for the same reason as that one, *primitive*.

CHAPTER III.

A Surface of the Second Degree and One of its Generators Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)\,p + yzq + z^2r,\ xp + yq,\ p + zq,$$

and leaves the surface of the second degree

$$y - zx = 0$$

with one of its generators, invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$ according to the formulae p. 310, our group has the form*

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4,$$
$$x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}_i\, x_ip_i.$$

§2.

Thus the transformations IO of the groups we now seek have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 = x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ S_3 = x_3p_2 - x_1p_4,$$
$$S_4 = x_3p_1 - x_2p_4,\ S_5 = x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}_i\, x_ip_i.$$

For the initial terms of these transformations the relations hold,

$$(S_1S_2) = 2S_1,\ (S_1S_3) = S_2,\ (S_2S_3) = 2S_3,\ (S_4S_5) = 2S_4,$$
$$(S_1S_4) = (S_1S_5) = (S_2S_4) = (S_2S_5) = (S_3S_4) = (S_3S_5) = 0,$$
$$(S_iU) = 0.$$

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* The form in the text is not that given immediately by the formulae p. *cit.*, but an equivalent one which we choose on account of the following calculations being so somewhat more convenient. We obtain the group given by the formulae p. *cit.*, when we perform the simple transformation

$$x_1' = x_3,\ x_2' = -x_1,\ x_3' = x_2,\ x_4' = x_4,$$

on the group in the text.

The same change is made in the next three chapters.

Thus we see that if U does not occur free, it can occur additively only with S_5, in form

$$S_5 + \alpha U, \qquad (\alpha \text{ const.}).$$

Let us see what transformations of an order higher than the first can occur.

No transformation IIO of the form

$$\xi_1^{(2)} p_1 + \xi_4^{(2)} p_4$$

can occur; for then we would have

$$\xi_1^{(2)} \equiv \xi_1^{(2)}(x_1 x_3), \quad \xi_4^{(2)} \equiv \xi_4^{(2)}(x_2 x_4),$$

and by combining with p_k, we find that then

$$\xi_1^{(2)} \equiv \xi_4^{(2)} \equiv 0.$$

Thus if a transformation IIO of form

$$Xf \equiv \sum_1^4{}^i \xi_i^{(2)} p_i$$

occurs, we must have either $\xi_2^{(2)} \neq 0$, or $\xi_3^{(2)} \neq 0$. We see also

$$\xi_2^{(2)} \equiv \xi_2^{(2)}(x_2 x_3), \quad \xi_3^{(2)} \equiv \xi_3^{(2)}(x_2 x_3), \quad \xi_1^{(2)} \equiv \xi_1^{(2)}(x_1 x_3 x_4), \quad \xi_4^{(2)} \equiv \xi_4^{(2)}(x_1 x_2 x_4).$$

Thus if

$$Yf \equiv \sum_1^4{}^i \xi_i^{(s)} p_i$$

is a transformation sO which occurs, by combining Yf properly with $x_2 p_3 - x_4 p_1$, we can arrange so that

$$x_2^s p_2 + \eta_3^{(s)} p_3 + \eta_1^{(s)} p_1 + \eta_4^{(s)} p_4$$

must occur, when we suppose $\xi_2^{(s)} \neq 0$. In this case thus, when s is the maximum order, we must have $s < 3$.

If $\xi_2^{(s)} = 0$, then $\xi_3^{(s)} \neq 0$, and

$$x_3^s p_3 + \eta_2^{(s)} p_2 + \eta_1^{(s)} p_1 + \eta_4^{(s)} p_4$$

must occur. Thus, as above, $s < 3$.

Thus, at all events, no transformation of an order higher than the second can occur.

What transformations IIO can occur in our group?

(1). Suppose

$$\sum_1^4{}^i \xi_i^{(2)} p_i$$

is a transformation *IIO* which occurs, and let $\xi_2^{(2)}$ be $\neq 0$. Then as above a transformation

$$Xf \equiv x_2^2 p_2 + \xi_3(x_2 x_3) p_3 + \xi_1(x_1 x_3 x_4) p_1 + \xi_4(x_2 x_4) p_4$$

must occur. Combine Xf with $x_1 p_1 - x_2 p_2$, and we see

$$Yf \equiv x_2^2 p_2 + x_2 \frac{\partial \xi_3}{\partial x_2} p_3 - \left(x_1 \frac{\partial \xi_1}{\partial x_1} - \xi_1\right) p_1 + x_2 \frac{\partial \xi_4}{\partial x_2} p_4$$

occurs. Either Yf is the same transformation as Xf, or not; if not, a transformation *IIO* of the form

$$\eta_3(x_2 x_3) p_3 + \eta_1(x_1 x_3 x_4) p_1 + \eta_4(x_2 x_4) p_4$$

which we get by subtracting Yf from Xf, must occur. Here we must have $\eta_3 \neq 0$, and by combining the last transformation properly with $x_2 p_3 - x_4 p_1$, we see a transformation of form

$$Zf \equiv x_2^2 p_3 + \bar{\eta}_1(x_1 x_3 x_4) p_1 + \bar{\eta}_4(x_2 x_4) p_4$$

occurs.

By combining p_3 with Zf we see that $\bar{\eta}_1 = \bar{\eta}_1(x_1 x_4)$.

Combine Zf with $x_3 p_3 - x_4 p_4$, and we see

$$x_2^2 p_3 + x_4 \frac{\partial \bar{\eta}_1}{\partial x_4} p_1 + \left(x_4 \frac{\partial \bar{\eta}_4}{\partial x_4} - \bar{\eta}_4\right) p_4$$

must occur. Subtract this transformation from Zf and we find

$$x_4 \frac{\partial \bar{\eta}_1}{\partial x_4} = \bar{\eta}_1, \quad x_4 \frac{\partial \bar{\eta}_4}{\partial x_4} = 2\bar{\eta}_4,$$

or $$\bar{\eta}_1 = \alpha_1 x_1 x_4, \quad \bar{\eta}_4 = \beta_4 x_4^2, \qquad (\alpha_1, \beta_4 \text{ const.}).$$

Thus $$Zf \equiv x_2^2 p_3 + \alpha_1 x_1 x_4 p_1 + \beta_4 x_4^2 p_4.$$

By combining Zf with p_1 and p_4 we find

$$\alpha_1 = \beta_4 = 0.$$

Thus $$Zf = x_2^2 p_3,$$

and no such transformation can occur.

Thus we find that Yf must be the same transformation as Xf; that is,

$$x_2 \frac{\partial \xi_3}{\partial x_2} = \xi_3, \quad x_1 \frac{\partial \xi_1}{\partial x_1} = 0, \quad x_2 \frac{\partial \xi_4}{\partial x_2} = \xi_4.$$

Thus $$\xi_3 = a x_2 x_3, \quad \xi_1 = \xi_1(x_3, x_4), \quad \xi_4 = b x_2 x_4,$$

or $$x_2^2 p_2 + a x_2 x_3 p_3 + (c x_3^2 + d x_3 x_4 + e x_4^2) p_1 + b x_2 x_4 p_4$$

occurs.

By combining this transformation with p_3, p_4, p_2 respectively, and expressing the results linearly in the transformation IO, we find when U occurs free,

$$c = e = 0; \; a = b = 1 = -d.$$

Thus in this case the transformation IIO,

$$X_1 f \equiv x_2^2 p_2 + x_2 x_3 p_3 - x_3 x_4 p_1 + x_2 x_4 p_4$$

occurs. Combine this transformation with $x_3 p_2 - x_1 p_4$ and we find that

$$X_2 f \equiv x_3^2 p_3 + x_2 x_3 p_2 + x_1 x_2 p_1 - x_1 x_2 p_4$$

also occurs, and by combining the general transformation IIO with p_k we find that $X_1 f$ and $X_2 f$ are the only transformations IIO which can occur when U occurs free. We easily see that if U does not occur, no transformation IIO can occur at all, and if U occurs in the form

$$S_5 + \alpha U \qquad (\alpha \text{ const.})$$

we find that $X_1 f$ and $X_2 f$ can occur where

$$\alpha = -2.$$

Now we come to case

(2), where $\xi_2^{(2)} \equiv 0$ in the original transformation IIO,

$$Xf \equiv \sum_1^4 {}_i \, \xi_i^{(2)} p_i.$$

In this case we have a transformation of form

$$\eta_3 p_3 + \eta_4 p_4 + \eta_1 p_1,$$

and we saw on the last page that no such transformation could occur.

In finding our groups we shall now distinguish the two cases where transformations IIO occur, and where no transformations IIO occur.

§3.

Suppose no transformations IIO occur.

If the transformation U occurs free, we normalize as usual with it, and find without difficulty the group

$$\boxed{\begin{array}{l} p_1, \; p_2, \; p_3, \; p_4; \; x_4 p_1 - x_2 p_3, \; x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4, \; x_3 p_2 - x_1 p_4, \\ x_3 p_1 - x_2 p_4, \; x_1 p_1 - x_2 p_2 - x_3 p_3 + x_4 p_4, \; x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4. \end{array}}$$

But this group is imprimitive, since the family of ∞^2 manifoldnesses

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

is invariant.

Suppose U does not occur free. Our transformations are then

$$p_k;\ x_4p_1 - x_2p_3 \equiv S_1,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2,\ x_3p_2 - x_1p_4 \equiv S_3,$$
$$x_3p_1 - x_2p_4 \equiv S_4,$$

$$S_5 + \alpha . U \equiv x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 + \alpha . \sum_{1}^{4}{}^{i}\, x_ip_i \equiv T,$$

where α is some constant.

Let us normalize with S_2.

We easily see that we can, without loss of generality, choose the transformations zero O, so that

$$\begin{cases} (p_1S_2) = p_1, \\ (p_2S_2) = -p_2, \\ (p_3S_2) = p_3, \\ (p_4S_2) = -p_4, \end{cases}$$

and we find without any difficulty, in the usual manner, by means of Jacobi's Identity, that all the relations (p_kS_i) and (p_kT) are normal. We wish to see how the transformations zero O are connected among each other.

We find directly from forming Jacobi's Identity with p_i, p_k and S_2,

$$\begin{cases} (p_1p_2) = b_2S_2 + bT + b_4S_4, \\ (p_1p_3) = b_1S_1, \\ (p_1p_4) = a_2S_2 + aT + a_4S_4, \\ (p_2p_3) = \beta_2S_2 + \beta T + \beta_4S_4, \\ (p_2p_4) = \gamma_3S_3, \\ (p_3p_4) = c_2S_2 + cT + c_4S_4, \end{cases}$$

where the $b_2b \ldots . c_4$ are certain constants.

Now form Jacobi's Identity with p_1, p_2, S_1, thus

$$((p_1p_2)\, S_1) + (p_1p_3) \equiv 0,$$

or
$$-2b_2S_1 + b_1S_1 \equiv 0,\ b_1 = 2b_2.$$

Proceed in exactly the same manner with the other p_i, p_k, S_8 and p_i, p_k, T, and we find a number of relations between our constants, viz:

$$a = b = c = a_4 = a_2 = \beta_2 = 0.$$
$$\alpha . b_1 = 0,\ \beta_4(\alpha - 2) = 0,\ \beta(\alpha - 1) = 0,\ c_4(\alpha - 1) = 0,\ b_4(\alpha - 1) = 0;$$
$$b_1 = \gamma_3 = 2b_2,\ b_4 = \beta = -c_4,\ b_2 = c_2.$$

(1). *Suppose* $\alpha = 1$, then $b_1 = \gamma_3 = b_2 = \beta_4 = 0$, and we have

$$(p_1p_2) = b_4S_4,\ (p_2p_3) = b_4T,\ (p_3p_4) = -b_4S_4,$$

and by forming Jacobi's Identity with p_1, p_2, p_3, we find $b_4 = 0$.

Hence all of our transformations are connected by normal relations in this case, and we find the group

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4,\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4,\ x_1p_1 + x_4p_4. \end{gathered}}$$

This is a subgroup of the last one found, and is imprimitive for the same reason that that one is.

(2). *Suppose* $\alpha \neq 1$.

(a). *Let* $\alpha = 2$; hence

$$b_1 = \gamma_3 = b_2 = c_2 = c_4 = b_4 = \beta = 0.$$

All relations are normal except

$$(p_2p_3) = \beta_4S_4.$$

Thus we see that p_1, p_4 is an invariant G_2 in our group, and hence the group is imprimitive.*

(b). *Suppose* $(\alpha - 1)(\alpha - 2) \neq 0$; hence

$$\beta = c_4 = b_4 = \beta_4 = 0,$$

and we have

$$\begin{aligned} (p_1p_2) &= b_2S_2, \\ (p_1p_3) &= 2b_2S_1, \\ (p_1p_4) &= 0 = (p_2p_3), \\ (p_2p_4) &= 2b_2S_3, \\ (p_3p_4) &= b_2S_2. \end{aligned}$$

---

* For

$$p_1 = 0,\ p_4 = 0$$

is an invariant complete system with the two solutions

$$x_2 = \text{const.},\ x_3 = \text{const.}$$

Thus our group leaves the family of ∞^2M_2, $x_2 = \text{const.}$, $x_3 = \text{const.}$, invariant.

Form Jacobi's Identity with p_1, p_2, p_3 and we see at once

$$b_2 = 0.$$

Hence all of our relations are in this case normal, and we find again the imprimitive group written above.

§4.

Suppose transformations IIO occur.

Let us further suppose—

(a). *U occurs free.* In this case we normalize with U as usual and find without difficulty the group:

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4; \; x_4p_1 - x_2p_3, \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \; x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4, \; x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4, \; x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \\ x_3 . U - (x_1x_2 + x_3x_4)p_4, \; x_2 . U - (x_1x_2 + x_3x_4)p_1. \end{array}}$$

This group is imprimitive, since the family of ∞^2 manifoldnesses,

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

is invariant. Let us now suppose

(b). *U does not occur free.* Here, as we saw above,

$$S_5 - 2U$$

must occur. Thus our transformations have the form

$$\begin{array}{l} p_k, \; x_4p_1 - x_2p_3 \equiv S_1, \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2, \\ \qquad x_3p_2 - x_1p_4 \equiv S_3, \; x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 \equiv S_4, \\ T \equiv - S_5 + 2U \equiv x_1p_1 + 3(x_2p_2 + x_3p_3) + x_4p_4, \\ \qquad\qquad X_1f \equiv x_3 . U - (x_1x_2 + x_3x_4)p_4, \; X_2f \equiv x_2 . U - (\;)p_1. \end{array}$$

Let us normalize with T. We have

$$(S_1T) = aX_1f + bX_2f, \qquad (a, b \text{ const.}).$$

Let us introduce a new S_1 by means of

$$\overline{S}_1 \equiv S_1 + AX_1f + BX_2f.$$

Then $$(\overline{S}_1T) = aX_1f + bX_2f - 3(AX_1f) + BX_2f).$$

Here we may choose our arbitrary constants A, B so that

$$(\overline{S}_1 T) = 0,$$

or

$$(S_1 T) = 0,$$

as we may write it.

Same way, we may without loss of generality choose the other S_k so that the relations $(S_k T)$ are all normal. The S_k and $X_i f$ are already connected by normal relations; we wish to find how the S_k are connected among each other. We have

$$(S_1 S_2) = 2S_1 + \alpha_1 X_1 f + \beta_1 X_2 f.$$

Form Jacobi's Identity with S_1, S_2 and T; thus

$$((S_1 S_2)\, T) + ((S_2 T)\, S_1) + ((T S_1)\, S_2) \equiv 0,$$

or

$$((S_1 S_2)\, T) \equiv 0,$$

i. e.

$$\alpha_1 = \beta_1 = 0.$$

Hence

$$(S_1 S_2) = 2S_1,$$

a normal relation. Analogously we find that all $(S_i S_k)$ are normal.

Now let us normalize the transformations zero O. We have

$$(p_1 T) = p_1 + \sum_{1}^{4}{}^{i} \alpha_i S_i + \alpha T + \beta X_1 f + \gamma X_2 f.$$

Introduce a new p_1 by means of

$$\overline{p}_1 \equiv p_1 + \sum_{1}^{4}{}^{i} A_i S_i + A T + \sum_{1}^{2}{}^{k} A'_k X_k f.$$

Thus

$$(\overline{p}_1 T) \equiv \overline{p}_1 - \sum_{1}^{4}{}^{i} A_i S_i - A T - \sum_{1}^{2}{}^{k} A_k X_k f + \sum_{1}^{4}{}^{i} \alpha_i S_i + \alpha T + \beta X_1 f$$

$$+ \gamma X_2 f - 2A_4 S_4 - 3 \sum_{1}^{2}{}^{k} A'_k X_k f.$$

Here we may choose the constants so that

$$(\bar{p}_1 T) = \bar{p}_1,$$

or

$$(p_1 T) = p_1.$$

Analogously

$$\begin{cases} (p_2 T) = 3p_2, \\ (p_3 T) = 3p_3, \\ (p_4 T) = p_4. \end{cases}$$

Now to find how the p_k are connected with the other transformations of our group. We have

$$(p_1 X_1 f) \equiv S_4 + a_1 X_1 f + a_2 X_2 f.$$

Form Jacobi's Identity with p_1, $X_1 f$ and T; thus

$$((p_1 X_1 f)\, T) + ((X_1 f T) p_1) - ((p_1 T)\, X_1 f) \equiv 0,$$

or

$$-2S_4 - 3\,(a_1 X_1 f + a_2 X_2 f) + 2\,(p_1 X_1 f) = 0,$$

i. e.

$$(p_1 X_1 f) = S_4.$$

Analogously we find that the other $(p_k X_i f)$ are also normal.

Now for the $(p_i S_k)$. We have

$$(p_1 S_1) = \sum_{1}^{4}{}^{i}\, \alpha_i S_i + \alpha T + \sum_{1}^{2}{}^{k}\, \beta_k X_k f.$$

Form Jacobi's Identity with p_1, S_1, T, thus

$$((p_1 S_1)\, T) - (p_1 S_1) = 0,$$

i. e.

$$(p_1 S_1) = 0.$$

In the same manner we find that the other $(p_k S_i)$ are normal also.

It remains to see how the transformations zero O are connected among each other. We have

$$(p_1 p_2) = \sum_{1}^{4}{}^{i}\, a_i p_i + \sum_{1}^{4}{}^{k}\, b_k S_k + bT + \sum_{1}^{2}{}^{j}\, c_j X_j f.$$

Form Jacobi's Identity with p_1, p_2, T, thus

$$((p_1 p_2)\, T) - 4\,(p_1 p_2) = 0,$$

i. e.

$$(p_1 p_2) = 0.$$

Analogously we find $(p_i p_k) = 0$.

Thus all of our transformations are connected by normal relations, and we find the group:

$$\boxed{\begin{array}{l} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4, \\ x_1p_1 + 3(x_2p_2 + x_3p_3) + x_4p_4,\ x_3 . U - (x_1x_2 + x_3x_4)p_4,\ x_2 . U - (x_1x_2 + x_3x_4)p_1 . \end{array}}$$

This is a subgroup of the one last found, and is imprimitive for the same reason that that one is.

Chapter IV.

A Surface of the Second Degree and Two Coinciding Generators Invariant in R_3.

§1.

The projective group written in xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,\ p + zq,$$

and leaves the surface $y - zx = 0$

and two of its generators which coincide invariant. When written linear and homogeneous in $x_1x_2x_3x_4$ our group has the form

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,\ x_3p_1 - x_2p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}_{i}\, x_i p_i .$$

§2.

Thus the transformations *IO* of the groups we seek in this chapter have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_2p_4,\ S_3 \equiv x_3p_2 - x_1p_4,\ S_4 \equiv x_3p_1 - x_2p_4,$$

to which may be added

$$U \equiv \sum_{1}^{4}{}_{i}\, x_i p_i .$$

As to the transformations of sO which may occur in our group, we find immediately as in Chapter III, §2, that when s is the maximum order,

$$s < 3.$$

Let us see what transformations IIO can occur.

The calculations here are similar to those in the above-mentioned chapter and §.

Suppose

$$\sum_{1}^{4} {}_i\, \xi_i^{(2)} p_i$$

is a transformation IIO which occurs, and let us suppose for the present that $\xi_2^{(2)} \neq 0$. Then we see a transformation IIO of form

$$Xf \equiv x_2^2 p_2 + \xi_3^{(2)}(x_2 x_3) p_3 + \xi_1^{(2)}(x_1 x_3 x_4) p_1 + \xi_4^{(2)}(x_2 x_4) p_4$$

must occur.

Combine Xf with S_2 and we find

$$Y \equiv x_2^2 p_2 + \left\{ x_2 \frac{\partial \xi_3}{\partial x_2} - x_3 \frac{\partial \xi_3}{\partial x_3} + \xi_3 \right\} p_3 + \left\{ x_2 \frac{\partial \xi_4}{\partial x_2} + x_4 \frac{\partial \xi_4}{\partial x_4} - \xi_4 \right\} p_4$$
$$- \left\{ x_3 \frac{\partial \xi_1}{\partial x_3} + x_1 \frac{\partial \xi_1}{\partial x_1} - x_4 \frac{\partial \xi_1}{\partial x_4} - \xi_1 \right\} p_1$$

must occur.

If Yf is not the same transformation as Xf, we see by subtraction that a transformation of the form

$$\eta_1^{(2)} p_1 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4$$

must occur, where $\eta_3^{(2)}$ is $\neq 0$. Hence we see that

$$Zf \equiv x_2^2 p_3 + \eta_1^{(2)}(x_1 x_4) p_1 + \eta_4^{(2)}(x_2 x_4) p_4$$

must belong to our group. Combine Zf and S_2, so

$$-3x_2^2 p_3 - \left\{ x_4 \frac{\partial \eta_1}{\partial x_4} - x_1 \frac{\partial \eta_1}{\partial x_1} + \eta_1 \right\} p_1 - \left\{ x_4 \frac{\partial \eta_4}{\partial x_4} + x_2 \frac{\partial \eta_4}{\partial x_2} - \eta_4 \right\} p_4$$

occurs. Multiply Zf by 3, and add to the last transformation; thus we get

$$2\eta_1 - x_4 \frac{\partial \eta_1}{\partial x_4} + x_1 \frac{\partial \eta_1}{\partial x_1} = 0, \quad 4\eta_4 - x_4 \frac{\partial \eta_4}{\partial x_4} - x_2 \frac{\partial \eta_4}{\partial x_2} = 0.$$

From the last equation
$$\eta_4 \equiv x_2^4 \bar{\eta}_4\left(\frac{x_4}{x_2}\right).$$

But since η_4 is only of the second degree, we have
$$\bar{\eta}_4 \equiv 0.$$

We now easily find by combining Zf with the p_k that also
$$\eta_1 \equiv 0.$$

Hence
$$Zf \equiv x_2^2 p_3,$$

and we see at a glance that this transformation cannot belong to our group.

Thus Yf must be the same transformation as Xf, *i. e.*
$$x_2 \frac{\partial \xi_3}{\partial x_2} - x_3 \frac{\partial \xi_3}{\partial x_3} = 0, \quad x_1 \frac{\partial \xi_1}{\partial x_1} + x_3 \frac{\partial \xi_1}{\partial x_3} - x_4 \frac{\partial \xi_1}{\partial x_4} = 0, \quad x_2 \frac{\partial \xi_4}{\partial x_2} + x_4 \frac{\partial \xi_4}{\partial x_4} = 2\xi_4.$$

Hence
$$\xi_3 = a x_2 x_3, \quad \xi_1 = b x_1 x_4 + c x_3 x_4, \quad \xi_4 = d x_2 x_4.$$

Thus
$$Xf \equiv x_2^2 p_2 + a x_2 x_3 p_3 + (b x_1 x_4 + c x_3 x_4) p_1 + d x_2 x_4 p_4.$$

Combine Xf with the p_k and we find at once
$$a = b = c = d = 0.$$

Since $x_2^2 p_2$ cannot occur, we see that no transformation *IIO* can occur at all.

Thus in the case where $\xi_2^{(2)}$, in the transformation *IIO*,
$$\sum_1^4 {}_i\, \xi_i^{(2)} p_i,$$

is $\neq 0$, no transformation *IIO* can occur in our group.

If $\xi_2^{(2)} = 0$, we have a transformation of the form
$$\eta_1^{(2)} p_1 + \eta_3^{(2)} p_3 + \eta_4^{(2)} p_4,$$

which, as we saw above, cannot occur.

Hence no transformations of an order higher than the first can occur in our groups.

We shall, as usual, distinguish the two cases where U occurs free and where U does not occur free.

§3.

Suppose U occurs free.

We normalize as usual with this transformation and find without any difficulty the group:

$$\boxed{\begin{array}{c} p_1 p_2 p_3 p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{array}}$$

This group is evidently imprimitive.

§4.

Suppose U does not occur free.

We easily see that in this case U can occur additively only in the form

$$S_4 + \alpha . U \qquad (\alpha \text{ const.}).$$

Thus our transformations IO are

$$S_1 \equiv x_4p_1 - x_2p_3,\quad S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\quad S_3 \equiv x_3p_2 - x_1p_4,$$
$$T \equiv S_4 + \alpha . U \equiv x_3p_1 - x_2p_4 + \alpha . \sum_1^4{}^i x_ip_i.$$

Let us normalize with S_2. We easily find that we may without loss of generality choose the transformations zero O, so that

$$\begin{cases} (p_1S_2) = p_1, \\ (p_2S_2) = -p_2, \\ (p_3S_2) = p_3, \\ (p_4S_2) = -p_4. \end{cases}$$

Now we wish to see how the p_k are connected with the S_i and with T. We have

$$(p_1S_1) \equiv \sum_1^3{}^i m_iS_i + mT.$$

Form Jacobi's Identity with p_1, S_1 and S_2, thus

$$((p_1S_1)\,S_2) - 3\,(p_1S_1) = 0,$$

i. e.

$$(p_1S_1) = 0.$$

Thus p_1 and S_1 are connected by a normal relation; in the same manner we find that the other $(p_k S_i)$ and $(p_k T)$ are normal.

It remains to see how the transformations zero O are connected among each other.

In the usual manner, by forming Jacobi's Identity with p_i, p_k and S_2, we find

$$\begin{cases} (p_1 p_2) = b_2 S_2 + b\,T, \\ (p_1 p_3) = b_1 S_1, \\ (p_1 p_4) = a_2 S_2 + a\,T, \\ (p_2 p_3) = \beta_2 S_2 + \beta\,T, \\ (p_2 p_4) = \gamma_3 S_3, \\ (p_3 p_4) = c_2 S_2 + c\,T. \end{cases}$$

Now form Jacobi's Identity with p_1, p_2 and S_1, thus

$$((p_1 p_2)\, S_1) + (p_1 p_3) = 0,$$

or

$$-\,2b_2 S_1 + b_1 S_1 = 0, \quad b_1 = 2b_2.$$

Proceed in the same manner with p_i, p_k, S_j; p_i, p_k, T and p_i, p_k, p_j; then we find that our above constants must satisfy the relations

$$b_1 = \gamma_3 = b_2 = c_2 = a_2 = \beta_2 = a = b = c = 0,$$
$$\alpha . \beta = 0.$$

Hence

$$(p_1 p_2) = (p_1 p_3) = (p_1 p_4) = (p_2 p_4) = (p_3 p_4) = 0;$$
$$(p_2 p_3) = \beta . T.$$

These relations show that the transformations p_1, p_4 form an invariant G_2 in the groups we here seek. Hence are these groups *imprimitive.**

If $\beta \neq 0$, it is easy to see that we find the following very peculiar imprimitive group:

$$\boxed{\begin{aligned} &p_1,\ p_2,\ p_3 + x_2 x_3 p_1 - \frac{x_2^2}{2} p_4,\ p_4 x_3 p_1 - x_2 p_4,\ \left(x_4 - \frac{x_2^2 x_3}{2}\right) p_1 - x_2 p_3 \\ &+ \frac{x_2^3}{6} p_4,\ x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4,\ x_3 p_2 - x_1 p_4 + \frac{x_3^3}{3} p_1. \end{aligned}}$$

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* Compare p. 329, note.

If, on the contrary, $\beta = 0$, our transformations are connected by normal relations, and we evidently get the imprimitive group:

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_3p_1 - x_2p_4 + \alpha\,(x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4). \end{array}}$$

CHAPTER V.

A Surface of the Second Degree and Two Different Generators of one Family Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ xp + yq,\ (xz - y)p + yzq + z^2r,\ yq + zr,$$

and leaves the surface of the second degree

$$y - zx = 0$$

and two different generators of one family on this surface invariant.

When written linear and homogeneous in $x_1x_2x_3x_4$, the group has the form

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4;\ x_3p_2 - x_1p_4;\ x_1p_1 - x_2p_2 - x_3p_3 - x_4p_4,$$

to which may be added

$$U \equiv \sum_1^4{}_i\, x_ip_i.$$

§2.

Thus the transformations *IO* of the groups we seek now have the forms

$$x_4p_1 - x_3p_3 \equiv S_1,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4 \equiv S_2,\ x_3p_2 - x_1p_4 \equiv S_3,$$
$$x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 \equiv S_4,$$

to which may be added

$$U \equiv \sum_{1}^{4} {}_i\, x_i p_i .$$

What transformations IIO can occur? If any transformation IIO occurs, it must have the form

$$\begin{aligned} &\xi_1^{(2)}(x_1 x_4)p_1 + \xi_2^{(2)}(x_2 x_3)p_2 + \xi_3^{(2)}(x_2 x_3)p_3 + \xi_4^{(2)}(x_1 x_4)p_4 \\ &\equiv (a x_1^2 + b x_1 x_4 + c x_4^2)p_1 + (\ )p_2 + (\ )p_3 + (\ )p_4 . \end{aligned}$$

By combining this transformation with the p_k we easily see that all the constants ab .... are zero.

Thus, no transformation of an order higher than the first can occur in our groups.

We shall distinguish as usual the cases where U occurs free and where U does not occur free.

§3.

U occurs free.

Here we normalize with the transformation U as usual and find without difficulty the group:

$$\boxed{\begin{array}{c} p_1,\ p_2,\ p_3,\ p_4;\ x_4 p_1 - x_2 p_3,\ \ x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4,\ \ x_3 p_2 - x_1 p_4, \\ x_1 p_1 - x_2 p_2 - x_3 p_3 + x_4 p_4,\ \ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 . \end{array}}$$

We see at once that this group is imprimitive since the family of $\infty^2 M_2$,

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

as well as that of

$$\begin{cases} x_1 = \text{const.} \\ x_4 = \text{const.} \end{cases}$$

are invariant.

§4.

U does not occur free.

It is easy to see that U can occur additively only in the form

$$S_4 + \alpha \,.\, U, \qquad (\alpha = \text{const.}).$$

Our transformations IO are

$$S_1 \equiv x_4p_1 - x_2p_3, \quad S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \quad S_3 \equiv x_3p_2 - x_1p_4,$$
$$S_4 + \alpha . U \equiv x_1p_1 - x_2p_2 + x_3p_3 + x_4p_4 + \alpha . \sum_1^4 {}^i x_ip_i \equiv T.$$

Let us normalize with S_2. We find as usual without trouble

$$\left\{\begin{aligned} (p_1S_2) &= p_1, \\ (p_2S_2) &= -p_2, \\ (p_3S_2) &= p_3, \\ (p_4S_2) &= -p_4. \end{aligned}\right.$$

We find in the ordinary manner that all the other relations (p_kS_i), (p_kT) are normal.

We wish to see how the transformations of the zero O are connected. We find in the usual manner by forming Jacobi's Identity with p_i, p_k, and S_2,

$$\left\{\begin{aligned} (p_1p_2) &= b_2S_2 + bT, \\ (p_1p_3) &= b_1S_1, \\ (p_1p_4) &= a_2S_2 + aT, \\ (p_2p_3) &= \beta_2S_2 + \beta T, \\ (p_2p_4) &= \gamma_3S_3, \\ (p_3p_4) &= c_2S_2 + cT. \end{aligned}\right.$$

By forming further Jacobi's Identity with p_i, p_k, S_j; p_i, p_k, T and p_i, p_k, p_j, we find for our above constants the relations

$$a = a_2 = \beta = \beta_2 = 0, \quad c = -b, \quad b_1 = \gamma_3 = 2b_2 = 2c_2, \quad b_2 - b + \gamma_3 = 0,$$
$$\alpha . b = \alpha . b_2 = 0.$$

If $\alpha \neq 0$, we see that all constants are zero, and our transformations are all connected by normal relations. This gives the group:

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4; \; x_4p_1 - x_2p_3; \; x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4; \; x_3p_2 - x_1p_4; \\ x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4 + \alpha\,(x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4). \end{gathered}}$$

If, on the contrary, $\alpha = 0$, we have

$$\begin{aligned} (p_1p_2) &= b_2(S_2 + 3S_4), \\ (p_1p_3) &= 2b_2S, \\ (p_1p_4) &= 0; \\ (p_2p_3) &= 0, \\ (p_2p_4) &= 2b_2S_3, \\ (p_3p_4) &= b_2(S_2 - 3S_4). \end{aligned}$$

Here we may suppose $b_2 \neq 0$, otherwise we would get the above group again. We may arrange so that $b_2 = 1$ if we put $b_2 . p_2$ for a new p_2, and $b_2 . p_3$ for a new p_3. This gives the group:

$$\boxed{\begin{array}{l} p_1,\ p_2 + 2x_1^2p_1 - (4x_1x_2 - 2x_3x_4)p_2 - 2x_1x_3p_3 + 2x_1x_4p_4, \\ p_4,\ p_3 + 2x_1x_4p_1 - 2x_2x_4p_2 - (4x_3x_4 + 2x_1x_2)p_3 + 2x_4^2p_4, \\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4, \\ x_1p_1 + x_2p_2 - x_3p_3 + x_4p_4. \end{array}}$$

Both the groups in this § are imprimitive, since the family of ∞^2 manifoldnesses

$$\begin{cases} x_1 = \text{const.} \\ x_4 = \text{const.} \end{cases}$$

is invariant under both.

CHAPTER VI.

A Surface of the Second Degree and all Generators of One Family Invariant in R_3.

§1.

The projective group written in the variables xyz has the form

$$r + xq,\ yq + zr,\ (xz - y)p + yzq + z^2r,$$

and leaves the surface of the second degree

$$y - zx = 0$$

and each generator, of one family, of this surface invariant.

Written linear and homogeneous in the variables $x_1x_2x_3x_4$ our group has the form

$$x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ x_3p_2 - x_1p_4,$$

to which may be added

$$U \equiv \sum_1^4 {}^i\, x_ip_i .$$

§2.

Thus the transformations IO of the groups we now seek have the forms

$$S_1 \equiv x_4p_1 - x_2p_3,\ S_2 \equiv x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4,\ S_3 \equiv x_3p_2 - x_1p_4,\ U \equiv \sum_1^4 {}^i\, x_ip_i,$$

where U may or may not occur. For the initial terms of these transformations the relations hold,

$$(S_1S_2) = 2S_1,\ (S_1S_3) = S_2,\ (S_2S_3) = 2S_3,\ (S_iU) = 0 .$$

Thus we see that if U does not occur free, it cannot occur additively.

Let us see what transformations IIO can occur.

We easily see that the general transformation IIO has the form

$$\begin{aligned} Xf &\equiv \xi_1^{(2)}(x_1x_4)p_1 + (\xi_2^{(2)}(x_2x_3)p_2 + \xi_3^{(2)}(x_2x_3)p_3 + \xi_4^{(2)}(x_1x_4)p_4 \\ &\equiv (a_1x_1^2 + bx_1x_4 + cx_4^2)p_1 + (\ )p_2 + (\ )p_3 + (\ )p_4, \end{aligned}$$

where a, b, c . . . . are certain constants. If now we combine Xf with p_1, p_2, p_3, p_4 and express the results linearly in the transformation IO, we find

$$a = b = c = \ldots . = 0 .$$

Thus no transformation IIO can occur, and so none of an order higher than the first can occur.

As usual, we have two cases according as U occurs or not.

§3.

Suppose U occurs.

Here we normalize with U as in the preceding chapters, and without any difficulty find the group:

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4;\ x_4p_1 - x_2p_3,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4, \\ x_3p_2 - x_1p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 . \end{gathered}}$$

This group is imprimitive, since the family of $\infty^2 M_2$,

$$\begin{cases} x_1 = \text{const.} \\ x_4 = \text{const.} \end{cases}$$

as well as that of

$$\begin{cases} x_2 = \text{const.} \\ x_3 = \text{const.} \end{cases}$$

are invariant.

§4.

Suppose U does not occur.

Let us normalize here with S_2. Our transformations are

$$p_1, p_2, p_3, p_4;\ x_4 p_1 - x_2 p_3 \equiv S_1,\ x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4 \equiv S_2,\ x_3 p_2 - x_1 p_4 \equiv S_3.$$

We easily see that, just as in §3, Chap. I, we may choose, without loss of generality, the transformations zero O, so that

$$\begin{cases} (p_1 S_2) = p_1, \\ (p_2 S_2) = -p_2, \\ (p_3 S_2) = p_3, \\ (p_4 S_2) = -p_4. \end{cases}$$

Now let us see how the transformations zero O are connected with the other transformations IO. We have for example

$$(p_1 S_1) = \sum_1^3 {}_i\, \alpha_i S_i, \qquad (\alpha_i \text{ const.}).$$

Form Jacobi's Identity with $p_1 S_1$ and S_2, thus

$$((p_1 S_1)\, S_2) - 3\,(p_1 S_1) = 0,$$

or

$$(p_1 S_1) = 0.$$

Thus $(p_1 S_1)$ is normal; same way we find that the other $(p_k S_i)$ are also normal.

It remains to see how the transformations zero O are connected among each other. We have

$$(p_1 p_2) = \sum_1^4 {}_i\, a_i p_i + \sum_1^3 {}_j\, b_j S_j.$$

Form Jacobi's Identity with p_1, p_2 and S_2, thus

$$((p_1 p_2)\, S_2) \equiv 0,$$

or

$$a_i = b_1 = b_3 = 0$$

and

$$(p_1 p_2) = b_2 S_2.$$

Same way
$$\begin{cases}(p_1p_3)=a_1S_1, \\ (p_1p_4)=d_2S_2, \\ (p_2p_3)=c_2S_2, \\ (p_2p_4)=e_3S_3, \\ (p_3p_4)=f_2S_2. \end{cases}$$

Now form Jacobi's Identity with p_1, p_2 and S_3, thus
$$((p_1p_2)\,S_3)-(p_2p_4)=0,$$
or
$$2b_2S_3-e_3S_3=0,\ 2b_2=e_3.$$
Same way
$$((p_1p_2)\,S_1)+(p_1p_3)=0,$$
or
$$-2b_2S_1+a_1S_1=0,\ 2b_2=a_1.$$
Further
$$-(p_3(p_1p_2))-(p_1(p_2p_3))+(p_2(p_1p_3))\equiv 0,$$
or
$$-b_2p_3-c_2p_1-a_1p_3=0,\ b_2=-a_1,\ c_2=0.$$
Thus
$$b_2=a_1=e_3=c_2=0,$$
and we easily find in the same way that
$$d_2=f_2=0.$$

Thus all of our transformations are connected by normal relations, and we evidently get the group:

$$\boxed{p_1,\, p_2,\, p_3,\, p_4;\ x_4p_1-x_2p_3,\ x_1p_1-x_2p_2+x_3p_3-x_4p_4,\ x_3p_2-x_1p_4.}$$

This is a subgroup of the one found in the last §, and is imprimitive for the same reason that that one is.

The necessary limits of this article do not permit me to give at present the calculations in the four cases which remain of this problem. Suffice it to say that I have finished these somewhat long and difficult calculations, and have shown that no primitive groups, except those already known, exist in the space of four dimensions. I hope to publish the calculations themselves at a later date, thereby furnishing a solution of a considerable part of the important and extensive problem of *finding* ALL *groups, primitive and imprimitive, in* R_4.

Synopsis of all Primitive Groups in R_4.

We shall designate those groups, which Lie has determined, as he has done, and refer to his works for an explanation of the names, when such is needed.

(A). *Groups which transform the directions through a point, of general position in R_4, which we hold, in the most general manner possible.*

1).
$$\boxed{p_k;\ x_i p_k;\ x_1 U;\ x_2 . U;\ x_3 . U,\ x_4 . U;\ i,\ k = 1 \ldots . 4.}$$

Here as usual U denotes $\sum_1^4{}^i x_i p_i$. This is the *general projective group* in R_4.

2).
$$\boxed{p_k,\ x_i p_k;\ i,\ k = 1 \ldots . 4.}$$

This is the *general linear group.*

3).
$$\boxed{p_k,\ x_i p_k,\ x_i p_k - x_k p_k;\ i \neq k;\ i,\ k = 1 \ldots . 4.}$$

This is the *special linear group.*

(B). *Groups which leave a Surface of the Second Degree in R_3 invariant.*

1).
$$\boxed{p_i,\ x_i p_k - x_k p_i;\ i,\ k = 1 \ldots . 4.}$$

This is the group of *Euclidean motions* (*Euklidische Bewegungen*).

2).
$$\boxed{p_i,\ x_i p_k - x_k p_i,\ \sum_1^4{}^j x_j p_j;\quad i,\ k = 1 \ldots . 4.}$$

This is the group of *Euclidean motions and similar transformations* (*Euklidische Bewegungen und Ähnlichkeitstoff*).

3).
$$\boxed{p_i - x_i . U,\ x_i p_k - x_k p_i;\quad i,\ k = 1 \ldots . 4.}$$

This group leaves a surface of second degree

$$\sum_1^4 {}_i\, x_i^2 = 1$$

in R_4 invariant.

4).

$$\boxed{p_i,\ x_i p_k - x_k p_i,\ U,\ 2x_i U - p_i \cdot \sum_1^4 {}_i\, x_i^2; \qquad i, k = 1 \ldots . 4.}$$

This is the group of *conform transformations.*

(C). *Groups which leave a twisted curve IIIO in R_3 invariant.*

1).

$$\boxed{\begin{gathered} p_k,\ x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3),\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4. \end{gathered}}$$

2).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 + 2x_1p_2 + 3x_2p_3,\ 2x_2p_1 + x_3p_2 + 3x_1p_4, \\ x_1p_1 - x_2p_2 + 3\,(x_4p_4 - x_3p_3). \end{gathered}}$$

(D). *Groups which leave a linear complex in R_3 invariant.*

1).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_1 - x_2p_2,\ x_2p_1,\ x_1p_2,\ x_3p_1 + x_2p_4, \\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_3p_3 - x_4p_4,\ x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4. \end{gathered}}$$

2).

$$\boxed{\begin{gathered} p_k,\ x_4p_1 - x_2p_3,\ x_4p_2 + x_1p_3,\ x_4p_3,\ x_1p_2,\ x_1p_1 - x_2p_2,\ x_2p_1, \\ x_3p_1 + x_2p_4,\ x_3p_2 - x_1p_4,\ x_3p_4,\ x_1p_1 - x_2p_2 + x_3p_3 - x_4p_4. \end{gathered}}$$

These are all the primitive groups in space of four dimensions: only eleven in number.